General Relativity Fall 2018
Lecture 24: Penrose diagrams, Kerr black holes

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PENROSE DIAGRAMS

Our goal here is to have a visual representation of an infinite spacetime with a finite coordinate range, while preserving its causal structure.

Penrose diagram of the Minkowski spacetime

The Minkowski metric in spherical polar coordinates is

$$ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2. \tag{1}$$

We change variables to

$$u \equiv t - r, \quad v \equiv t + r, \quad -\infty < u \leq v < +\infty. \tag{2}$$

$$ds^2 = -du dv + \frac{1}{4}(v-u)^2 d\Omega^2, \tag{3}$$

Now define

$$U \equiv \arctan(u), \quad V \equiv \arctan(v), \quad -\pi/2 < U \leq V < \pi/2. \tag{4}$$

$$du = \frac{dU}{\cos^2(U)}, \quad v - u = \tan(V) - \tan(U) = \frac{\sin V \cos U - \sin U \cos V}{\cos U \cos V} = \frac{\sin(V-U)}{\cos U \cos V} \tag{5}$$

$$ds^2 = (\cos U \cos V)^{-2} \left[-dU dV + \frac{1}{4}\sin^2(V-U) d\Omega^2 \right]. \tag{6}$$

Finally, define

$$T \equiv V + U, \quad R \equiv V - U, \quad 0 \leq R < \pi, \quad |T| < \pi - R, \tag{7}$$

$$\cos U \cos V = \frac{1}{2}(\cos T + \cos R) \equiv \frac{1}{2}\omega \tag{8}$$

$$\Rightarrow \boxed{ds^2 = \omega^{-2}\left[-dT^2 + dR^2 + \sin^2 R \, d\Omega^2\right]}. \tag{9}$$

The metric in parenthesis is that of a $\mathbb{R} \times S^3$, where S^3 is the 3-sphere. So the Minkowski metric is **conformally related to part of** $\mathbb{R} \times S^3$, hence has the same causal structure (same timelike, null, spacelike character of vectors, same null geodesics). Note that it is only *part* of that spacetime, since the time variable T is limited to $\pi - R$.

The full transformation is

$$T = \arctan(t+r) + \arctan(t-r), \qquad R = \arctan(t+r) - \arctan(t-r), \tag{10}$$

We show the Penrose diagram of Minkowski spacetime in Fig. 1, i.e. a representation of part of $\mathbb{R} \times S^3$ to which it is conformally related. Each point represents a 2-sphere. Light cones are at 90-degree angles in this conformal spacetime.

Let us discuss a few points of interest:

• i^+, i^i correspond to $r =$ constant, and $t \to \pm\infty$. They are called **future and past timelike infinity** respectively. These are actually points $(T = \pm\pi, R = 0)$.

• i^0 corresponds to $t =$ constant, $r \to +\infty$. It is called spatial infinity. It is a point $(T = 0, R = \pi)$.

• $\mathcal{I}^+, \mathcal{I}^-$ (pronounced "scri plus, scri minus"), correspond to $r - t =$ constant, $r + t \to +\infty$ and $r + t =$ constant, $r - t = -\infty$. They are the future and pass null infinity, respectively. These are null surfaces. All outgoing radial null geodesics end at $\mathcal{I}^+$. All incoming radial null geodesics started at $\mathcal{I}^-$.

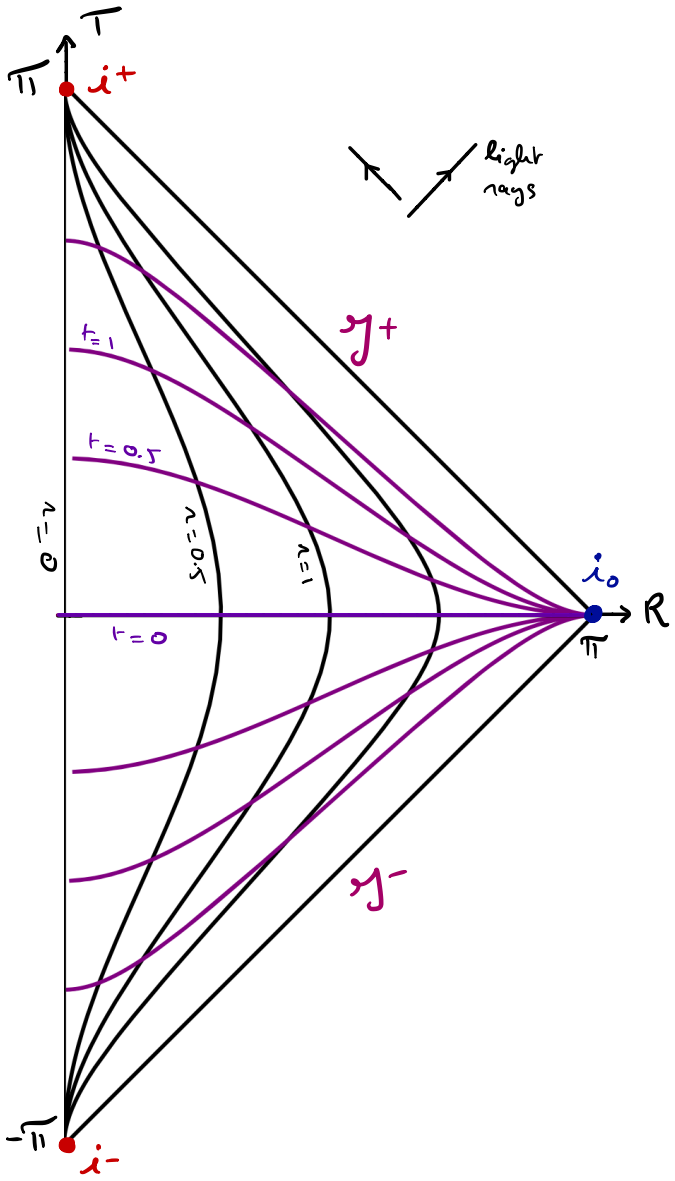

FIG. 1. Penrose diagram for Minkowksi spacetime

Schwarzschild metric

Recall that in Kruskal-Szekeres coordinates the Schwarzschild metric is

$$ds^2 = \tfrac{32M^3}{r}\mathrm{e}^{-r/2M}(-dT^2+dR^2)+r^2d\Omega^2, \qquad R\in\mathbb{R}, \quad T^2<1+R^2 \tag{11}$$

$$(r/2M-1)\mathrm{e}^{r/2M}\equiv R^2-T^2. \tag{12}$$

We focus on the (T,R) part only, and do a transformation similar to (10):

$$\tilde{T}=\arctan(T+R)+\arctan(T-R), \qquad \tilde{X}=\arctan(T+R)-\arctan(T-R). \tag{13}$$

This would be the same as Minkowski conformal diagram, except that $R\in\mathbb{R}$ (instead of $r\geq 0$) and $T^2-R^2<1$.

Using the following rule,

$$\tan(a+b)=\frac{\tan a+\tan b}{1-\tan a\tan b}, \tag{14}$$

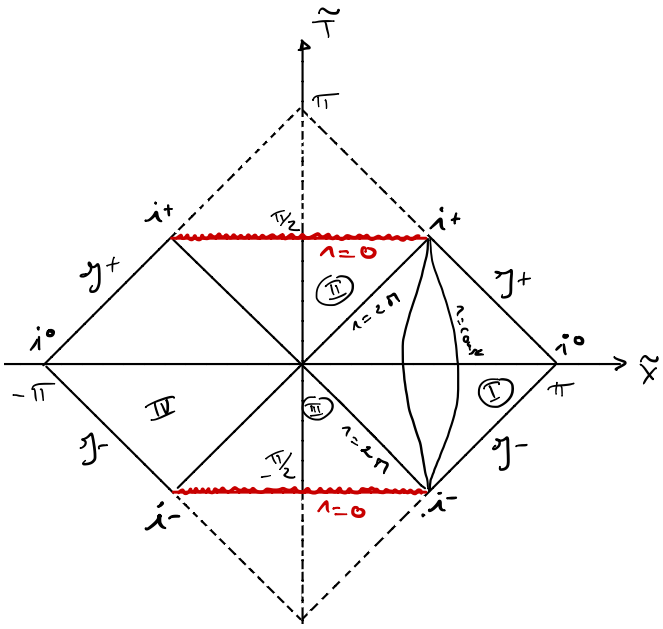

FIG. 2. Penrose diagram for the extended Schwarzschild metric

we find

$$\tan(\tilde{T}) = \frac{2T}{1+R^2-T^2}. \tag{15}$$

Therefore, the singularity $r=0$, which corresponds to $1+R^2-T^2 \to 0^+$, corresponds to $\tilde{T} \to \mathrm{sign}(T)\pi/2$.

We show the resulting Penrose diagram in Fig. 3. Light cones are also at 90-degree angles on this diagram. It has the same asymptotic structure as flat space time (future and past timelike infinities, spatial infinities, future and past null infinities), except it has 2 asymptically flat regions, as well as the singularities at $\tilde{T} = \pm\pi/2$.

This kind of representation is quite convenient. For instance, we can use it to represent the spacetime of a collapsing star.

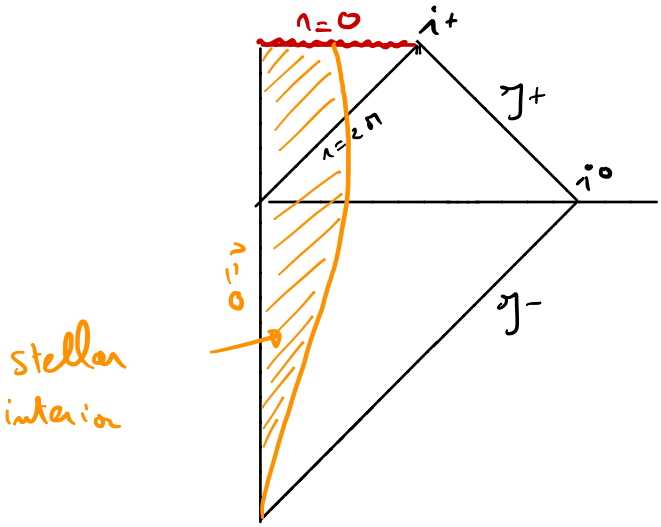

FIG. 3. Penrose diagram for a star collapsing into a Schwarzschild black hole

KERR BLACK HOLES

Just like the Schwarzschild solution represents a vacuum spacetime with a mass (as measured from e.g. Kepler's laws in the asymptotically flat regions), the Kerr solution represents a vacuum spacetime with a mass *and* angular momentum. Again, this is defined by the asymptotic form of the metric.

The Kerr metric, in Boyer-Lindquist coordinates is

$$ds^2 = -\frac{\rho^2}{\Sigma^2}\Delta dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta(d\varphi - \omega dt)^2, \tag{16}$$

$$\Delta \equiv r^2 - 2Mr + a^2, \quad \rho^2 \equiv r^2 + a^2\cos^2\theta, \quad \Sigma^2 \equiv (r^2+a^2)^2 - a^2\Delta\sin^2\theta, \quad \omega \equiv \frac{2aMr}{\Sigma^2}. \tag{17}$$

The parameter a has dimensions of mass (or length) and is such that $|a| < M$.

The inverse metric is given by

$$\mathbf{g}^{-1} = -\frac{\Sigma^2}{\rho^2\Delta}(\partial_t + \omega\partial_\varphi)^2 + \frac{\Delta}{\rho^2}\partial_r^2 + \frac{1}{\rho^2}\partial_\theta^2 + \frac{\rho^2}{\Sigma^2\sin^2\theta}\partial_\varphi^2. \tag{18}$$

For $a \to 0$, this metric reduces to Schwarzschild.

For $r \gg M, a$, we have

$$ds^2 \approx -(1-2M/r)dt^2 - 4\frac{aM}{r}\sin^2\theta dt d\varphi + dr^2 + r^2 d\theta^2. \tag{19}$$

We recognize the the second term as

$$\frac{4}{r^2}(\hat{x}\times\vec{J})\cdot d\vec{x}dt, \tag{20}$$

where $\vec{J} = aM\hat{z}$, where $\hat{z}$ is the polar direction used to define φ. Therefore the parameter a is **angular momentum per unit mass**.

This metric has **two Killing vector fields**, ∂_t and ∂_φ (but it does not have two more spacelike fields: it is only axially symmetric, but not spherically symmetric). As a consequence, $E = -u_t$ and $L = u_\varphi$ are conserved along geodesics. The (big) difference with Schwarzschild is that orbits are not "planar", i.e. we cannot rotate the coordinate system to place any given in the equatorial plane. Here the equatorial plane $\theta = \pi/2$ is indeed a special plane, in the sense that spacetime is symmetric with reflexion across this plane. So only orbits in the actual equatorial plane remain planar.

Equatorial orbits

Let us consider equatorial orbits, with $\theta = \pi/2$. Setting $g^{\mu\nu}u_\mu u_\nu = -1$, we get

$$-1 = -\frac{\Sigma^2}{\rho^2\Delta}(E - \omega L)^2 + \frac{\rho^2}{\Delta}\dot{r}^2 + \frac{\rho^2}{\Sigma^2}L^2, \tag{21}$$

where we used the fact that $g^{rr} = 1/g_{rr}$. Just like we did for Schwarzschild, we can write this equation as $\dot{r}^2 + V_{\text{eff}}(r) = \mathcal{E}$. The circular orbits are obtained for $V'_{\text{eff}}(r) = 0$, the stable ones are those with positive second derivative, and the unstable ones with negative second derivative. Those with $V''_{\text{eff}}(r) = 0$ are marginally stable. Just like for Schwarzschild, there is an innermost stable circular orbit (ISCO), whose radial coordinate satisfies the following equation (see e.g. C. Hirata's lecture notes, lecture 27):

$$r^2 - 6Mr + 8aM^{1/2}r^{1/2} - 3a^2 = 0. \tag{22}$$

This is a quartic equation for $r^{1/2}$. We recover $r_{\text{isco}} = 6M$ for $a \to 0$. For $a > 0$ (corresponding to prograde orbits, having fixed the sign of $L > 0$), the ISCO moves to smaller radii. For $a < 0$, corresponding to retrograde orbits, the ISCO moves outwards.

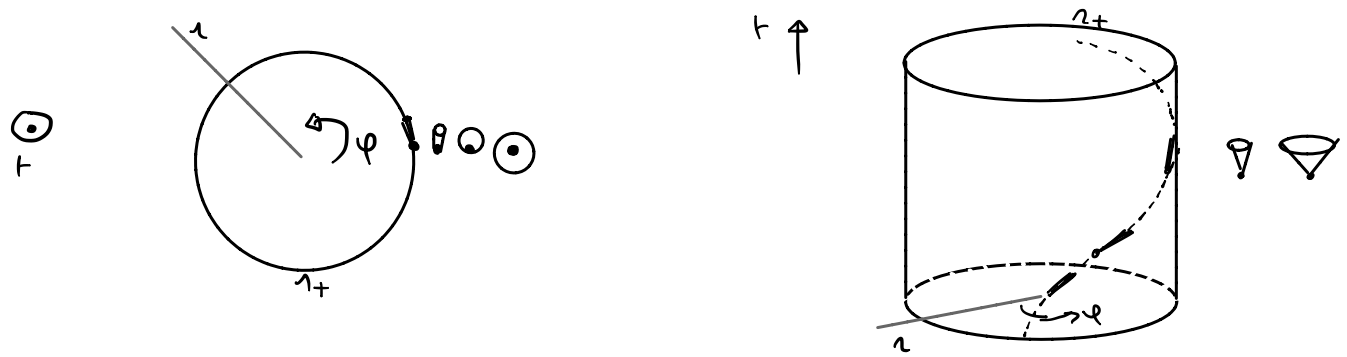

FIG. 4. Closing of light cones as $r \to r_+ = M + \sqrt{M^2 - a^2}$ in Boyer-Lindquist coordinates.

Frame dragging

Suppose a particle falls from infinity, starting at rest, i.e. $u_t = -1, u_\varphi = 0$. We then have

$$\frac{d\varphi}{dt} = \frac{u^\varphi}{u^t} = \frac{g^{\varphi t}u_t + g^{\varphi\varphi}u_\varphi}{g^{tt}u_t + g^{t\varphi}u_\varphi} = \frac{g^{\varphi t}}{g^{tt}} = \omega = \frac{2aMr}{\Sigma^2}. \tag{23}$$

So as the particle falls in, it acquires an angular velocity ω: this is a frame-dragging effect. Note that this is a meaningful statement as the Killing vector fields ∂_t and ∂_φ are "special" and tied to the symmetries of spacetime.

Light-cone structure, Kerr horizon

Consider null geodesics:

$$\left(\frac{dr}{dt}\right)^2 + \frac{\Sigma^2\Delta}{\rho^4}\sin^2\theta\left(\frac{d\varphi}{dt} - \omega\right)^2 + \Delta\left(\frac{d\theta}{dt}\right)^2 = \frac{\Delta^2}{\Sigma^2}. \tag{24}$$

The function Σ^2 is always greater $(r^2+a^2)^2 - a^2\Delta = r^4 + r^2a^2 + 2a^2Mr > 0$. The function Δ vanishes $r_\pm \equiv M \pm \sqrt{M^2 - a^2}$.

To visualize the light cone, consider equatorial orbits, with $\theta = \pi/2 =$ constant. The innermost and outermost edges of the cone, where $|dr/dt|$ is maximized, correspond to $d\varphi/dt = \omega$, $dr/dt = \pm\Delta/\Sigma$. Starting from $r \gg M$, the light cone is at 90-degree (spacetime is asymptotically flat). As $r \to r_+$, the light cone gets askew and closes, see Fig. 4.

This is similar to what happened for Schwarzschild. The apparent closing of light cones is merely due to the choice of coordinates. An observer falling towards r_+ will reach it and cross it in a finite proper time, even though it takes an infinite coordinate time. Seen from outside, such an infalling particle seems to never reach the hozirzon, and to spiral around forever.

Like in Schwarzschild, this is just a coordinate singularity: nothing dramatic actually happens at r_+. Nevertheless, it is a **horizon**, and nothing can escape it.

Ergosphere

For any particle, massive or massless, we have

$$g_{\mu\nu}p^\mu p^\nu \le 0. \tag{25}$$

Explicitly, this means

$$\frac{\Sigma^2}{\rho^2}\sin^2\theta 2\omega\dot\varphi\dot t \ge g_{tt}\dot t^2 + \frac{\rho^2}{\Delta}\dot r^2 + \rho^2\dot\theta^2 + \frac{\Sigma^2}{\rho^2}\sin^2\theta\dot\varphi^2 \ge g_{tt}\dot t^2. \tag{26}$$

After some algebra, we find

$$g_{tt} = -\frac{1}{\rho^2}\left(r^2 - 2Mr + a^2\cos^2\theta\right). \tag{27}$$

This becomes positive for $r < r_{\rm ergo} \equiv M + \sqrt{M^2 - a^2\cos^2\theta}$, which is always larger than the horizon $r+$ (except at the poles $\theta = 0, \pi$). So, inside the ergosphere, we must have $d\varphi/dt > 0$: frame dragging is so strong that any particle (massive, massless, geodesic or not), must have a positive $d\varphi/dt$.

The Penrose process

Suppose we drop a particle from a far distance, with conserved energy $E_{\rm in} = -p_t$. Now suppose we make sure that this particle decays into two particles inside the ergosphere. Conservation of 4-momentum implies, in particular, $E_{\rm in} = E^{(1)} + E^{(2)} \equiv -p_t^{(1)} - p_t^{(2)}$

Since the ergosphere is outside the horizon, it is possible for at least one of the particles to escape back to infinity, say particle 1. Suppose particle 2 plunges towards the horizon. Since ∂_t is spacelike inside the ergosphere and 4-momenta are timelike, it is in principle possible to arrange it so that $E^{(2)} = -p^{(2)}_\mu \partial^\mu_t < 0$ – ask yourself why this would not be possible in general, the answer is deeper than you may think! With this setup, we would then have $E_{\rm out} = E^{(2)} > E_{\rm in}$. So it is, in principle, possible to extract energy from a Kerr black hole!