General Relativitiy
Gravitational Waves

In empty space, the linearized Einstein equations are (with the harmonic coordinate conditions)

$$\Box \bar{h}_{ij} = 0 \tag{1}$$
$$\partial_i \bar{h}^i_j = 0 \tag{2}$$

Since these equations are linear, we can fourier tansform them and get that a solution for $\bar{h}_{ij}$ has the form $\bar{h}_{ij} e^{i\omega(t - \vec{n}\cdot\vec{x})}$ where $\vec{n}$ is the direction of propagation. We can assume without loss of generality, that the vector $\vec{n}$ is in the x direction. (we just rotate the coordinates). The second set of equations implies that all of the components must have the same ω and $\vec{n}$ since they relate all of the components to each other. Also h_{ij} will have the same form since the trace of $\bar{h}_{ij}$ will also have this form.

A small coordinate transformation $x^i \to x^i + \zeta^i$ affects h_{ij} as

$$h_{ij} \to h_{ij} + \partial_i \zeta_j + \partial_j \zeta_i \tag{3}$$

Since we want to keep the coordinates harmonic, we need

$$\Box \zeta^i = 0 \tag{4}$$

This will ensure that the second of the equations above is satisfied. We will also choose ζ^i so that its dependence is the same as h_{ij} namely $e^{i\omega(t-x)}$.

Now lets say that we have some solution. We still have a large set of coordinate transformations we can carry out. We will first use the freedom to set first h, the trace, equal to 0. Under the coordinate transformation we have that $h \to h + 2\partial_i \zeta^i$ or $h \to h + i\omega\zeta^0 - i\omega\zeta^1$ Thus, if we choose ζ such that $\zeta^0 - \zeta^1 = -h/(2i\omega)$, we will have set h after the transformation equal to 0. Let us therefor assume that we have done this, and $h = 0$ so $\bar{h}_{ij} = h_{ij}$, and also $\partial_i {h^i}_j = 0$. (This choice or ζ will obey $\Box\zeta^i = 0$ and thus this preserves the harmonic condition.

Now, with $h = 0$, we need to set $h_{01} = 0$ while preserving the tracelessness of h_{ij} which means we need ζ_i to obey

$$(\partial_0 \zeta^0 - \partial_1 \zeta^1) = i\omega(\zeta_0 + \zeta_1) = 0 \tag{5}$$
$$(\partial_t \zeta_1 + \partial_x \zeta_0) = i\omega(\zeta_0 - \zeta_1) = -h_{01} \tag{6}$$

from which we have $\zeta_0 = -\zeta_1 = -h_{01}/(2i\omega)$. Ie, by making this coordinate transformation, we now have $h = h_{01} = 0$. We now make a coordinate tranformation so as to set $h_{02} = 0$. To do so we do not want to have a ζ_0 component since that would upset the conditions we have already set. Thus, we want

$$\zeta_2 = -h_{02}/(i\omega) \tag{7}$$

since $h_{02} \to \partial_t \zeta_1 + \partial_z \zeta_0$. This clearly does not upset the other conditions we have already set. Similarlty, we choose

$$\zeta_3 = -h_{03}/i\omega \tag{8}$$

to set h_{03}=0. We have thus set all of $h = h_{01} = h_{02} = h_{03} = 0$

But we have

$$\partial_t h^0{}_0 + \partial_x h^1{}_0 + \partial_y h^2{}_0 + \partial_z h^3{}_0 = 0 \tag{9}$$

or recalling that we have used coordinate transformatons to set all of $h^a{}_0 = -h_{a0} = 0$ (recalling that $\eta_{aa} = -1$ we thus have that h_{00} must also be zero.

We also have

$$-\partial_t h^0_a + \partial_x h^1{}_0 + \partial_y h^2{}_0 + \partial_z h^3{}_a = 0 \tag{10}$$

where a=1,2,3, or $i\omega h_{1a} = 0$ Thus the only terms remaining are $h_{22},\ h_{23},\ h_{33}$ as bbeing non-zero. But since$h = h_{22} + h_{33}$ is also zero, the only components of the linearized metric that are potentially non-zero are $h_{33} = -h_{22}$ an $h_{23} = h_{32}$. These are the two polarizations of the linearized metric. By rotating y, z around the x axis, one can transform one of these polarizations into the other. Looking at the first of these, this implies that when h_{22} is larger than 0 (ie the distance between two particles lying on the y axis is larger than it was without the gravity wave), then h_{33} is smaller and vice versa.

Thus the metric looks like

$$ds^2 = dt^2 - dx^2 - (1 + h_{22})dy^2 - (1 - h_22)dz^2 \tag{11}$$

If we look at geodesics of this metric we have

$$\begin{aligned} \frac{d}{ds}((1 + h_{22})\frac{dy}{ds}) = 0 \\ \frac{d}{ds}((1 + h_{22})\frac{dz}{ds}) = 0 \end{aligned} \tag{12}$$

which means that if the particle starts out with no motion in the y or z diretions, the gravity wave will not change that. They will remain at rest in these directions.

Varying with respect to t and x we get

$$\begin{aligned}
\frac{d^2t}{ds^2} + \partial_t h_{22}((\frac{dy}{ds})^2 - (\frac{dz}{ds})^2) = 0 \\
\frac{d^2t}{ds^2} + \partial_x h_{22}((\frac{dy}{ds})^2 - (\frac{dz}{ds})^2) = 0
\end{aligned} \tag{13}$$

again, if the particle is at rest in the y and z directions, there is zero acceleration in the t an x directions so if it starts out at rest in the x direction as well, it stays at rest. Ie, particles at rest in this coordinate system stay at rest when the gravity wave goes by. Note that although they do not move, the distances between two of the particles in the y and z directions do change even though the particles do not move under geodesic motion.

If they do not move, how can you detect whether or not a gravitational wave has come by? The answer is via the change in distances. Although they do not move the distances beween particles do change. If those particles are part of a solid, then the atoms in the solid want to be a certain distance apart (due to Quantum mechanics– eg the electrons in the H atom want to be 53 picometers away from the nucleus, uniformly in alldirections, Similarly in a solid the atoms want to be a certain distance apart from each other. As the gravity wave goes by it changes those distances, in shear like fashion. The gravity wave does not change the volume, but does change the shape. Most solids will react if they are disturbed, and in this case it is the shear modulus , not the compressibility, or bulk modulus, that comes into play. When the atoms feel their shape distored they react by exerting forces on each other which starts the atoms moving. It is this reaction of the atoms on each other when the gravity wave distorts their shape that starts the material moving. And that motion may well continue as the wave goes by.

To understand the difference between shear and bulk modulus, think of the solid called Jello. This is highly incompressible. It has the compressibility of water, about $2 \cdot 10^9 N/m^2$ (atmospheric pessure is of the order of $10^5 N/m^2$). A change in volume of water of about .01% will create a pressue of about 1 atmosphere. On the othe hand, Jello has an extremely small shear modulus. A very small pressure will distort the shape by a large amount. Jello would make an extremely poor gravitational wave detector.

The first gravity wave detectors were the so called bar detectors. The idea was essentially invented by Joe Weber at the U Maryland. Machining lumps of aluminium of about a tonne in mass(or later niobium or other metals) one looked for changes in the shape. In particular one hoped to get resonance effects in which the frequency of the gravity wave was just equal to a natural resonant frequency of the bar, so as to produce a large effect on the bar. He then instrumented the bar with strain gauges, which would be sensitive to the change in the shape of the bar as it vibrated.

The gravity wave does not move the detector. Rather it changes the shape of the detector, a changing shape which one hopes to detect. (even a soure at the center of our galaxy in which a solar mass of energy were converted to gravity waves, would only change the shape by parts in about one part in 10^{16}

Weber began his work in the mid 60's, and began to claim that he saw signals. When Douglas at Syracuse and Amaldi in Rome got interested and built their own detetors, Weber claimed to see coincident detections between the detectors. He also claimed that his detections depended on the Siderial time (the day going from a star being staight ovehead to it being staight overhead the next day, about 23hr 56min later rather than the sun being straight overhead 24 hours later.) Unfortunately, once others got involved they persistantly saw nothing even though they built more sensitive detectors.

The other idea was to directly measure the distance between two "inertial" masses (massed tavelling along geodesics) Of course on earth the ground is accelerating upward (is no-geodesic) so one must ignore that and make them as geodesic as possible in the other directions.

in shape of the material during and after the passage of the gravity wave.

0.1 Sources of gravity waves.

we have shown that (except for terms which are independent of time) we can write the linearized equations such that only the spatial parts of the metric perturbations are non-zero. All of the other components can be set to zero by carrying out a harmonic coordinate transformation. In Harmonic coordinates, we have

$$\Box\bar{h}_{ij} = -16\pi T_{ij} \tag{14}$$

This has as solution

$$\bar{h}_{ij} = \int 4T_{ij}\frac{\delta(t-t'-|x-x'|)}{|x-x'|}d^3x'dt' \tag{15}$$

where we are interested only in i and j spatial components (the term is the standard Green's function for the relativistic wave equation which you have (should have?) seen in Electromagnetism). $|x-x'| = \sqrt{(\vec{x}-\vec{x}')\cdot(\vec{x}-\vec{x}')}$ is the distance between the two spatial points laveled by $x,\ x'$. To lowest order in $|x'|/|x|$ this is just

$$\bar{h}_{ij} = 4\int T_{ij}\frac{\delta(t-t'-|x|)}{|x|}d^3x'dt' = \int 4T_{ij}(t-|x|,x')d^3x'/|x|. \tag{16}$$

Since we are interested in purely the spatial components, let us write the indices as a, b, c in order to remind us of this.

Now, we can rewrite the integrand by recalling that $\partial_i x^a = \delta^a_b$. Thus $T_{ab} = T_{ac}\partial_a x^c$ and

$$\int T^{ab}d^3x' = \int T^{ac}\partial'_c x'^a = \int \partial' c(T^{ac}(t_r,x')x'^b) - \partial_c(T^{ac})x^b)d^3x' \tag{17}$$

But the first term is a complete integral and as long as the energy momentum tensor goes to 0 sufficiently far away, it is will be 0. The second term is the spatial divergence of T and from the conservation of T we have $\partial_c T^{ac} = \partial_0 T^{a0}x'^b$. Similarly, we can now write

$$\begin{aligned}\int T^{a0}x'^b d^3x' &= \int T^{d0}x'^b\partial_d x'^a d^3x' \qquad (18)\\ &= \int \partial_d(T^{do}x'^a x'^b d^3x' - \int(\partial_d T^{d0})x'^b x'^a - T^{d0}(\partial_d x'^b)x'^a d^3x'\end{aligned}$$

Because T is symmetric, the last term is the same as the term on the left hand side. The first term is again 0 because of the finiteness of the energy distribution, and the second term can again be written in terms of the time derivative. Thus

$$\int T^{ab}d^3x' = \frac{1}{2}\partial_t^2\int T^{00}x'^a x'^b d^3x' \tag{19}$$

The right hand side is just the second time derivative quadrapole moment of the energy distribution. Thus the gravitational radiation emitted is proportional to the second derivative of the quadrapole moment of the energy distribution.

$$\bar{h}^{ab} = 2\partial_t^2 Q^{ab}(t - |x|)/|x| \tag{20}$$

This obeys the wave equation in polar coordinates

$$(\partial_t^2 - \frac{1}{r^2}\partial_r r^2 \partial_r)(f(t-r)/r) = 0 \tag{21}$$

where $r = |x|$. One can therefor again carry out a coordinate transformation to set the trace equal to 0, and the components of h orthogonal to the radial vector to 0, just as for plane waves, to get

$$h^{ab} = 2\partial_t^2 Q_{TT}^{ab} \tag{22}$$

where TT refers to the transverse traceless part of the tensor Q (ie,

$$Q_{TT}{}_a^a = 0 \tag{23}$$

$$Q_{TT}{}_{ab} x^b = 0 \tag{24}$$

0.2 Non-linear plane waves

If one demands that one can have a couple of spatial translation Killing vectors, one can find the an exact solution to the vacuum waves which represents an exact solution for a plane wave.

$$ds^2 = (\ddot{a}(u)(x^2 - y^2) + \ddot{b}(u)xy)du^2 + 2dudv - dx^2 - dy^2 \tag{25}$$

This solution obeys the harmonic condition

$$g = -1 \tag{26}$$

$$\partial_i \sqrt{|g|} g^{ij}) = 0 \tag{27}$$

$$\begin{aligned} g^{uu} &= 0 \qquad (28)\\ g^{uv} &= g^{vu} = 1 \\ g^{vv} &= -(\ddot{a}(u)(x^2 - y^2) + \ddot{b}(u)xy) \\ g^{xx} &= g^{yy} = -1 \end{aligned}$$

with alli others being 0. The only term which has any spatial dependence is g^{vv} and we also have that $\partial_v g^{vv} = 0$

Defining

$$\Gamma_{ijk} = g_{kl}\Gamma^l_{ij} = \frac{1}{2}(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij}) \tag{29}$$

which means that the only non zero terms must have two u plus one of of u,x or y.

$$\begin{aligned} \Gamma_{uuu} &= \frac{1}{2}(\ddot{a}(u)(x^2+y^2) + 2\ddot{b}(u)xy) \\ \Gamma_{uux} = -\Gamma_{uxu} &= -\frac{1}{2}\partial_x g_{uu} = -x\ddot{a}(u) - 2\ddot{b}(u)y \\ \Gamma_{uuy} = -\Gamma_{uyu} &= \frac{1}{2}(y\ddot{a}(u) - 2\ddot{b}(u)x) \end{aligned} \tag{30}$$

and

$$\begin{aligned} \Gamma^v_{uu} &= \frac{1}{2}(\ddot{a}(u)(x^2+y^2) + 2\ddot{b}(u)xy) \\ \Gamma^x_{uu} &= x\ddot{a}(u) + \ddot{b}(u)y \\ \Gamma^y_{uu} &= -y\ddot{a}(u) + \ddot{b}(u)x \\ \Gamma^v_{ux} &= x\ddot{a}(u) + \ddot{b}(u)y \\ \Gamma^v_{uy} &= -y\ddot{a}(u) + \ddot{b}(u)x \end{aligned} \tag{31}$$

Because $det(g) = 1$, none of the terms which are derivatives of g survive and in particulare $\gamma^i_{ji} = 0$. The other term is of the form $\Gamma^k_{il}\Gamma^l_{jk}$ and is zero because if k is v, making the first term non-zero, then the k in the second term gives a Γ which is 0. Similarly is k is x, then then l must be v in the second term and there is no nonzero Γ which has $l = v$ in the first term. the other terms i R_ij containing the second derivatives of g_{ij} are zero because the equations solve the linearized equations.

$$R_i j = \partial_k \Gamma^k_{ij} - \partial_i \Gamma^k_{kj} - \partial_j \Gamma^k_{ki} + \partial_k \Gamma^k_{ji} \tag{32}$$

where the second and third terms contain derivatives of g which are zero, and the first and fourth must have $i = j = u$ and k eitehr x or y, or

$$R_{uu} = -(\partial_x(g^{xx}\partial_x + \partial_y g^{yy}\partial_y)g_{uu} = 0 \tag{33}$$

Thus the Einstein tensor for this metric is 0, so this is a fully non-linear solution to the source free Einstein equations. It is also a linerized solution which obeys the harmonic condition. Also

$$\partial_i \eta^{ij} \partial_j h_{kl} = 0 \tag{34}$$

with $h_{uu} = (\ddot{a(u)}(x^2 - y^2) + 2\ddot{b}(u)xy)$ the only non-zero term, and all others 0.

This metric does not look like a plane wave, uniform in the x y directions. However, one can make a coordinate transformation to show that it is. I will go backward, assuming that $b(u)$ is zero. (I have not been able to find the explicit transformation which gives both $a(u)$ and $b(u)$ non-zero. A solution with $a(u)$ non-zero can be converted to one with $a(u)$ zero and $b(u)$ non-zero by rotating x,y by 45 degrees.)

Consider the metric

$$ds^2 = d\tilde{v}du - f(u)^2 d\tilde{x}^2 - g^2(u)d\tilde{y}^2 \tag{35}$$

Now define $x = f(u)\tilde{x}; \quad y = f(u)\tilde{y}$ so the metric becomes

$$ds^2 = d\tilde{v}du - (dx - \frac{\dot{f}}{f}xdu)^2 - (dy - \frac{\dot{g}}{g}ydu = -\left((\frac{\dot{f}}{f})x^2 + (\frac{\dot{g}}{g})^2y^2\right)du^2 + du(d\tilde{v} + 2\frac{\dot{f}}{f}xdx + 2\frac{\dot{g}}{g}y$$

Take $\tilde{v} = v - \frac{\dot{f}}{f}x^2 - \frac{\dot{g}}{g}y^2$ so that

$$d\tilde{v} = dv - 2\frac{\dot{f}}{f}xdx + 2\frac{\dot{g}}{g}ydy - \frac{\dot{\overline{\dot{f}}}}{f}x^2 + \frac{\dot{\overline{\dot{g}}}}{g}y^2 \tag{37}$$

and

$$ds^2 = -\left(\frac{\dot{\overline{\dot{f}}}}{f}x^2 + \frac{\dot{\overline{\dot{g}}}}{g}y^2 + (\frac{\dot{f}}{f})^2x^2 + (\frac{\dot{g}}{g})^2y^2\right)du^2 + dudv - dx^2 - dy^2 \tag{38}$$

Thus the two metrics are equivalent if

$$-\frac{\ddot{f}}{f} = \frac{\ddot{g}}{g} = \ddot{a} \tag{39}$$

If $f \approx 1$, then

$$f = 1 - a + \int\int \ddot{a} a du du + O(a^3) \tag{40}$$

$$g = 1 + a(t) + \int\int \ddot{a} a du du + O(a^3) \tag{41}$$

Now

$$\int\int \ddot{a} a du du = (\frac{1}{2}a^2 - \int\int (\dot{a})^2 du du) \tag{42}$$

If $a(u)$ is non zero only for a finite period in u, then at late times a and a^2 are zero, but $\int\int(\dot{a})^2 dudu = \int(\dot{a})^2 du(u+u0)$ and both f and g go to zero in some finite u. Ie, distance between lines of constant $\tilde{x}$ and $\tilde{y}$ go to 0. This looks like a singularity, and was what led Einstein to say that the non-linear equations for gravity waves were inconsistant. However, it is clear from the u, v, x, y coordinates that the metric is perfectly regular, even when f, g go to 0 (where $\ddot{a} = 0$).

The geodesic equations are

$$\frac{d}{ds}(f^2(u)\frac{dx}{ds}) = 0 \tag{43}$$

$$\frac{d}{ds}(g^2(u)\frac{dy}{ds}) = 0 \tag{44}$$

which means that if the particles start with zero velocity their velocity will remain 0. However, the distance between two particles with different values of x or y will decrease as f or g The particles are focused by the gravity wave. This is true independent of what the z velocity of the particles is, as long as u keeps increasing along the path of the particle.

Now, we know that matter focuses light which if one passes just at the surface of the ball, through an angle of $4M/r$ where r is the radius of the ball of matter. The amount of matter between two particles at location $x0$ and $x0 + 2\Delta$ is $\pi\Delta^2 T\rho$ where ρ is the matter density, and T is the thickness of the slice. Thus $4M/r = 4\pi\rho\Delta^2/\Delta = \Delta/L$ where L is the distance before the two light rays come together. If the gravity wave is a narrow pulse, centered at u=0. Then f will go to 0 in a value of $1 = \delta u \int /dota^2 d\hat{u} \approx \Delta T \dot{a}^2$ This give $\rho = \frac{\dot{a}^2}{4\pi}$ as the mass density of the gravitational radiation pulse, or to lowest order in a or in $(f-1)$, the amplitude of the gravity wave.

The gravitatonal waves carries energy which gravitates in that it attracts matter and light just as matter does.

0.3 Flat spacetime

In the region outside the wave (a=0) , the spacetime has the metric (since $f = g = 1 - \kappa^2 u$ to lowest order, or, taking the origin for u as $1/kappa$, $f = g = -u$ to give the metric

$$ds^2 = d\tilde{u}dv - \tilde{u}^2(dx^2 + dy^2) \tag{45}$$

This metric is singular at $\tilde{u} = 0$. It also flat spacetime because the curvature tensor $R_{ijkl} = 0$.

But in u, v, x, y coordinates, the metric is just

$$ds^2 = dudv - dx^2 - dy^2 \tag{46}$$

which is just flat spacetime in double null coordinates, or, setting $u = t - z; \quad \tilde{v} = t + z$ we get

$$ds^2 = dt^2 - dz^2 - dy^2 - dx^2 \tag{47}$$

which is just flat spacetime in Minkowski coordinates.

Inserting into the transformation of v we get

$$v = (t + z) + \frac{\tilde{x}^2 + \tilde{y}^2}{t - z} \tag{48}$$

or

$$0 = -(t - z)v + t^2 - z^2 - \tilde{x}^2 - \tilde{y}^2 = (t - v/2)^2 - (z - v/2)^2 - \tilde{x}^2 - \tilde{y}^2 \tag{49}$$

Thus, the surfaces of constant v are just null cones whose apex lies along the surface $t - z = (v/2 - v/2) = 0$ and $x = y = 0$, which is a u=0 null line.

The flat slices on the intersection of the v=constant and u=const are the intersection of the the light rays converging to the point$v = \text{const}, \tilde{x} = \tilde{y} = 0$ and the null plane defined by $u = const$. At any time t, this intersection is the intersection of the surface of the sphere which is the collapsing light beam ($v = v_0$) and the plane which is the plane $u = u_0$. That intersection is a circle centered on $\tilde{x} = \tilde{y} = 0$, and $z = t - u_0$. The radius of the circle is

$$R = \sqrt{(t - v_0/2)^2 - (t - u_0 - v_0/2)^2} = \sqrt{2tu_0 - u_0^2 - u_0\tilde{v}_0} \tag{50}$$

Ie, the surface of intersection is a parabaloid of revolution. In the horizontal direction, the distances are the usual x-y distances. In the vertical direction, both t and z change by the same amount, and this is a null displacement so has zero lenth. Thus, although the parabola looks highly curved the distance between two points on the parabola is just the same as the distance between the two points projected down to a constant z surface.

This is the flat metric on the side of the gravity wave after the collision between a plane and the gravity wave. The gravity wave focuses the original plane into a sphere which collapses down to a point.

It is interesting that the light which crosses the sheet of gravity wave travelling in the opposite direction to the gravity wave gets focused, while the gravity wave itself does not focus itself. Ie, two beams of gravity waves travelling parallel to each other do not affect each other. They do not attrace or repell each other.

Penrose has studied the solution when two sheets of gravity waves cross each other. In that case each focuses the other, and the plane in which they focus together becomes a true, not just a coordinate singularity. This is not surprizing since one is focusing a whole infinite sheet of gravity waves down to a point.

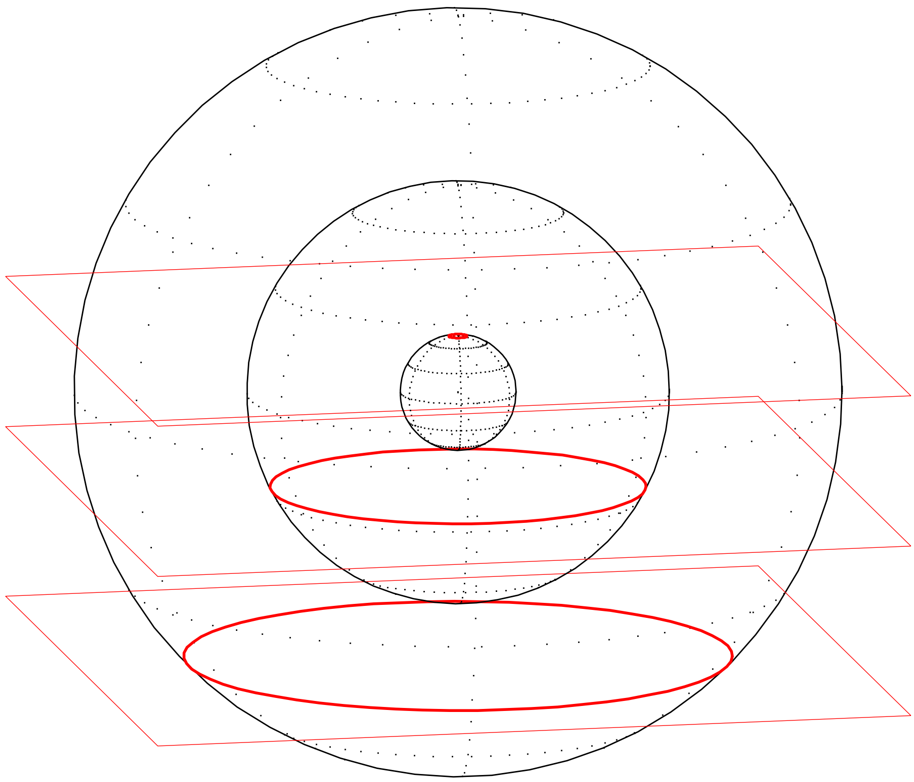

Figure 1: