Physics 530-11
Numerical technique

0.1 ADM spacetime

Einstein's equations are horrendously complicated and non-linear. Thus, except for highly symmetric situations, finding exact solutions is almost impossible. Furthermore, determining the effects of non-linearities on the solutions of the equations becomes very difficult. A perturbative solution is hard, not least because there is no "small parameter" to expand the solution in. The coupling constant G_N, the Newtonian constant, is not dimensionless, nor is there any way to make it so. For electromgnetism $\frac{e^2}{\hbar c}$ is dimensionaless and small (about 1/137), and one can think about doing an expansion in terms of it. For gravity, G_N is dimensionful, and one cannot combine it with the other constants of nature (like $\hbar$ or c to make a dimensionless quantity. The best one can do is to get dimensionful quantities, like the Plank length ($\sqrt{\frac{G_N\hbar}{c^3}}$ is a length, divided by c, a time, etc). None of these is dimensionless. So there is nothing which one could use as a universal expansion parameter.

Einstein's equations link the metric through space and time. They are also coordinate invariant so they also do not produce unique solutions. Rather then created solution which one can change by doing a coordinate transformation.

The theory also has problems, like singularities. Thus, for example, the black hole has only a coordinate singularity at $r = 2M$, they have a genuine singularity (where the curvature in any coordinate system, goes to infinity at $r = 0$.) To solve these equations one can transfer them into the usual initial values, with the Einstein equations determining the time developement from these initial conditions. It was Arnowit, Deser and Misner who gave the almost universally accepted way of describing the equations as a combination of inital data plus temporal equations.

Let us assume that we have divided the spacetime into a sequence of surfaces of constant time. At present we have no idea what that means, except each of the surfaces is supposed to be a spacelike surface. That means that any curve which is restricted to lying within the surface itself has a tangent vector which is spacelike.

Note that because of the importance of spacelike vectors, one usually, as a matter of convenience, choses the metric to have signature of (-,+,+,+) so that a spacelike vector has a positive length squared. (We used the other convention, (+,-,-,-) previously because we were mainly interested in the motion of particles in the spacetime, and particle orbits are timelike.)

These spacelike surfaces are labeled by a parameter t, and points within the surface are labeled by three other coordinates, x^a. We will use indices a,b,c,d,e,.. as labeling these spatial coordinates.

For the metric inside this spacelike surface we will use $\gamma_{ab}(t,\mathbf{x})$ where $\mathbf{x}$ are the spatial coordinates.

The full metric will be of the form

$$ds^2 = g_{tt}dt^2 + 2g_{ta}dtdx^a + \gamma_{ab}dx^a dx^b \tag{1}$$

Now, let us relabel the timelike components as

$$g_{ta} = N_a \tag{2}$$

and call these the shift vectors, while we define

$$N^2 - \gamma^{ab}N_aN_b = -g_{tt} \tag{3}$$

where γ^{ab} is the inverse three dimensional metric to γ_{ab}. N is called the lapse function, and N_a the shift function. Coonsider the vector n^i which is supposed to be orthogonal to any vector inside the surface. Such a tangent vector to a curve in the surface will be of the form $\tau^0 = 0$, with t^a arbitrary. For the product with n_i to be zero, n_i will have only a 0 component, and $n^i = (g^{tt}n_0, g^{ta}n_0)$ a perpedicualr vector to the surface will have a displacement both in time and in spatial coordinate. Let us choose $n_t = dt$ the small coordinate distance between to nearby surfaces.

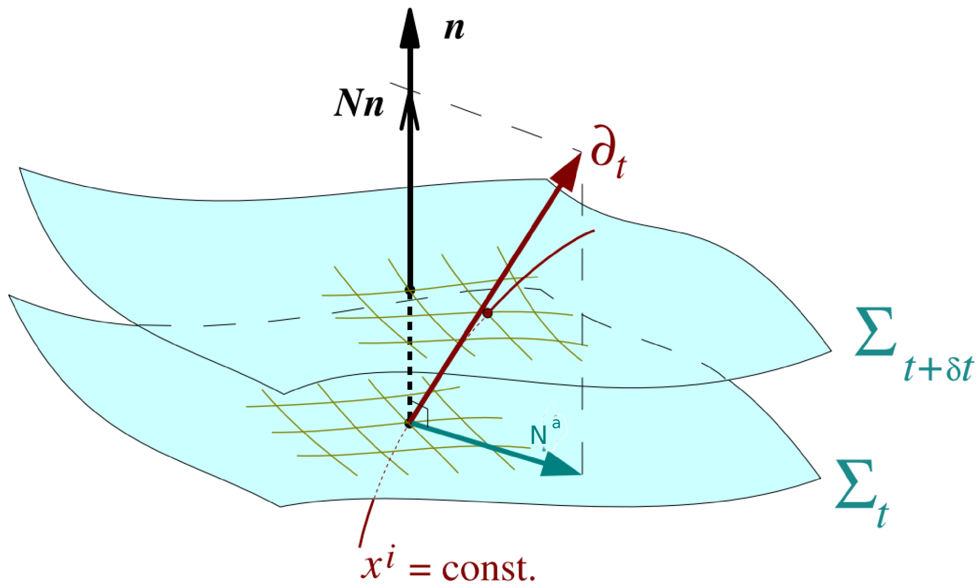
Let us first define the inverse metric to understand what is going on here.

$$g^{tt} \equiv g^{00} = -\frac{1}{N^2} \tag{4}$$

$$g^{ta}]equiv g^{0a} = g^{a0} = \gamma^{ab}\frac{N_b}{N^2} \tag{5}$$

$$g^{ab} = \gamma^{ab} - \frac{N^aN^b}{N^2} \tag{6}$$

$$g = det(\mathbf{g}) = N\det(\gamma) \tag{7}$$



(The last expression is one of the main reasons why we chose to define N as I did) Checking, we have

$$g^{0i}g_{i0} = g^{00}g_{00} + g^{0a}g_{0a} = (-\frac{1}{N^2})(-N^2 + N_aN^a) + N_a\frac{N^a}{N^2} = 1 \quad (8)$$

$$g^{0i}g_{ia} = g^{00}g_{0a} + g^{0b}g_{ba} = -\frac{1}{N^2}N_a + \frac{N^b}{N^2}\gamma_{ba} = 0 \quad (9)$$

$$g^{ai}g_{i0} = g^{a0}g_{00} + g^{ab}g_{b0} = \frac{N^a}{N^2}(-N^2 + N_cN^c) + (\gamma^{ab} - \frac{N^aN^b}{N^2}) = 0 \quad (10)$$

$$g^{ai}g_{ib} = g^{a0}g_{0b} + g^{ac}g_{cb} = \frac{N^a}{N^2}N_b + \gamma^{ac} - \frac{N^aN^c}{N^2})\gamma_{cb} = \delta^a_b \quad (11)$$

Ie, g^{ij} is the inverse of g_{ij}.

Or we can write

$$ds^2 = \gamma_{ab}(dx^1 + N^adt)(dx^b + N^bdt) - N^2dt^2 \quad (12)$$

Let us choose $n_i = \alpha(1, 0, 0, 0)$ a unit normal to the surface which goes perpendicularly from the surface t to the surface $t + dt$ where α is a normalising factor, so that $n_in_jg^{ij} = 1$ or $1 = \alpha^2g^{00} = -\frac{\alpha^2}{N^2}$. (this is assumed to be a timelike vector so with our current definition of the sign of the metric, it has length squared negative.) Thus we must have $\alpha = -N$ for this to be a unit vector and for the tangent vector n^i to point into the future. Consider the tangent vector to the curve at constant x^a from t to $t+dt$ which is $t^i = (dt, 0, 0, 0)$. The projection of t^i to n_i is $-(t^j n_j)n^i$ (the minus sign is so that the projection of the projection is just the projection). Now, the vector $t^i + t^j n_j n^i$ is perpendicular to n^i so must lie inside the surface. But this is $(dt, 0, 0, 0) + dtN^2(g^{00}, g^{0a}) = dt(0, N^a)$. Ie, the vector t^i is a combination of a perpendicualr vector to the surface t=const, plus a displacement along the surface given by N^a. This is called the shift vector, and the perpendicular component is called the Lapse.

WE can define a covariant derivative with respect to the three metric γ_{ab} by D_A so that

$$D_a V^b = \partial_a V^b + \frac{1}{2}\gamma^{bc}(\partial_a \gamma_{cd} + \partial_d \gamma_{ca} - \partial_c \gamma_{ad})V^d \tag{13}$$

Also, let us look at the extrinsic cuvature defined by

$$(\nabla_C n_D)(\delta_A^C + n^C n_A) = -K_{AB} \tag{14}$$

(The tensor

$$P_A^C = (\delta_A^C + n^C n_A) \tag{15}$$

is the projection into the surface. Consider any tangent vector k^i which lies inside the surface. It is orthogonal to n^A, the perpendicular to the surface, so $K^A n_A - 0$. And $k^A(\delta_A^C + n^C n_A) = k^C$, and $n^A(\delta_A^C + n^C n_A) = n^C + n^A n_A n^C = n^C - n^C = 0$.Ie, it projects any tangent vector into the surface.) The minus sign in the definition of $K_{ab}$is a convention, which is used for example by Misner, Thorn and Wheeler.

The tensor K_{AB} lies in the surface (is orthogonal to the unit vector which is orthogonal to the surface.) It essentially measures the amount by which the unit vector orthogonal to the surface splays out or in. If it splays in in some direction, then $k^A K_{AB} k^B$ is positive. It is called the extrinsic curvature because it measures how much the surface is curved in the 4 dimensional embedding space. It is also a symmetric tensor. because the projection of the antisymmetric part is zero. $P_C^A P_D^B(\nabla_A n_B - \nabla_B n_A)$. (in coordinates, the antisymmetric part of the derivative of n, $\nabla_i n_j - \nabla_j n_i = \partial_i n_j - \partial_j n_i$ has at least one component which is purely in the tempral direction since $n_i = N\delta_i^0$, which is killed by the projection.

For example a flat sheet of paper rolled into a cylinder has a flat internal metric (rolling it does not alter the internal distances) but has a non-zero extrinsic curvature.

The components are

$$\begin{aligned}
K_{ab} &= -\nabla_a n_b = \partial_a(N,0,0,0)_b - \Gamma^0_{ab}N & (16)\\
&= \frac{1}{2N}(\partial_t\gamma_{ab} - \partial_b N_a - \partial_a N_b) + \frac{N^c}{2N}(\partial_b\gamma_{ac} + \partial_b\gamma_{ac} - \partial_c\gamma_{ab}) & (17)\\
&= \frac{1}{2N}(\partial_t\gamma_{ab} - (D_a N_b + D_b Na)) & (18)
\end{aligned}$$

K_{ab} is a tensor in the 3 surface. and it includes the time derivative of the three metric.

The Hilbert action for General Relativity is

$$\mathcal{L} = \int \sqrt{|g|}Rd^4x \tag{19}$$

R looks like $g(\partial\Gamma - \Gamma\Gamma)$. The first term has second derivatives of the metric. but is linear in those and they are multiplied by metric components without derivatives. Ie, the second derivative terms in R look like $g\partial^2 g$. One can do an integration by parts in time (and selectively in space) so as to only have terms– which look like $\partial g\partial g$– quadratically in the first time derivative of the components of g. In particular, one finds that there are only time derivatives of γ and not of the N and N_a once on has done this. Ie, the N and N_a act like Lagrange multipliers, not like dynamical variables. While there are certainly spatial derivatives of both in the Lagrangian, there are no temporal ones.

One can write the equations in terms of the extrinsic curvature tensor, or one can write in terms of the momenta π^{ij}. In particular, taking the functional derivative of the Lagrangian with respect to $\partial_t\gamma_{ab}$ one gets the conjugate momenta to the metric

$$\pi^{ab} = \sqrt{\gamma}(-K^{ab} + K\gamma^{ab}) \tag{20}$$

where $K = \gamma^{ab}K_{ab}$ is the trace of extrinsic curvature and $\pi = \pi^{ab}\gamma_{ab}$ os the trace of the momentum.

Then

$$\mathcal{L} = \int \pi^{ij}\partial_t\gamma_{ij} - NH - N_a P^a \tag{21}$$

where one can take π^{ab}, γ_{ab} and N, N_a as the independent functions to be varied Here

$$\begin{aligned} H &= -\sqrt{\gamma}(\mathcal{R}) - \frac{1}{\sqrt{\gamma}}(\frac{1}{2}\pi^2 - \pi^{ab}\pi_{ab}) &(22)\\ H^a &= -2(\partial_a\pi^{ab} + \Gamma^b_{ac}\pi^{ac}) &(23)\end{aligned}$$

(π^{ab} is a tensor density since it has a $\sqrt{\gamma}$ already contained in its definition. If it were an ordinary tensor, one would have

$$D_a q^{ab} = \frac{1}{\sqrt{\gamma}}\partial_a\sqrt{\gamma}q^{ab} + \Gamma^b_{ac}a^{ac} \tag{24}$$

but since π already has the $\sqrt{\gamma}$ the covariant derivative does not require the exra term.)

The Hamiltonian is then just

$$\mathcal{H} = \int (NH + N_a H^a)d^3x \tag{25}$$

The variables to be varied are γ_{ab}, π^{ab}, N and N_a which are all components of the metric and the time derivative of the metric. If one varies with respect to N, N_a one gets

$$\begin{aligned} H &= 0 &(26)\\ H^a &= 0 &(27)\end{aligned}$$

as four of the equations of motion. Ie, if the equations of motion are satisfied, the Hamiltonian density is 0. (note that there may still be boundary terms which are not zero, if the spacetime has a boundary). If one solves the equations of motion, then $\mathcal{H}$ is 0. The energy on any solution is 0. o These four equations are constraint equations. They restrict the intial data (values of γ_{ab} and π^{ab}) that one can specify on the surface. This is similar to electromagnetism, where $\partial_a E^a = 0$ restricts the inital data for E^a one can choose.

One also does not get any independent equations for N and N_a and thus one can use any values one wants for these. H and H_a do not include time derivatives of these quantities. The intial data one would at first suspect should be γ_{ab} and π^{ab}. But the H and H^a are finctions of these quantities on the initial data surface. Thus one cannot arbitrarily assign these quantities. One has four constraints which would give four fewer initial data. Thus, instead of 12 initial data at each point in spacetime, one only has 12-4=8. Furthermore one can choose 4 functions as coordinate conditions. This leaves just 4 independent degrees of freedom– 2 metric and two momenta. These just correspond to the two second order wave degrees of freedom.

Varying these equations with respect to the momenta and the metric, we get the equations of motion.

$$\partial_t \gamma_{ab} = \frac{\delta \mathcal{H}}{\delta \pi^{ab}} \tag{28}$$

$$= \frac{2N}{\sqrt{\gamma}}(\pi_{ab} - \frac{1}{2}\pi g_{ab}) + D_a N_b + D_b N_a \tag{29}$$

$$\partial_t \pi^{ab} = -\frac{\delta \mathcal{H}}{\delta \gamma_{ab}} \tag{30}$$

$$= -N\sqrt{\gamma}\left(R^{ab} - \frac{1}{2}R\gamma\right) + \frac{N}{2\sqrt{\gamma}}\gamma^{ab}\left(\pi^{cd}\pi_{cd} - \frac{1}{2}\pi^2\right) - \frac{2N}{\sqrt{\gamma}}\left(\pi^{in}\pi_n{}^j - \frac{1}{2}\pi\pi^{ab}\right) \tag{31}$$

$$-\sqrt{\gamma}\left(D^a D^b N - \gamma^{ab} D^c D_c N\right) + D_c\left(\pi^{ab} N^c\right) - (D_c N^a)\pi^{cb} - (D_c N^b)\pi^{ca} \tag{32}$$

These are of course highly non-linear equations with loads of terms. But one can use numerical techniques to find solutions.

Two of the biggest problems in doing the numerical evolution were

a) Finding coordinate conditions.

b) handling the horizons

The first was battled with for many years. One of the biggest problems was that the coordiantes in general were not causal. Coordinate changes could travel much faster than light. It seems that this tended to cause instabilities in the evoltion. For at least 20 year people were unable to produce an evolution which was stable (ie did not crash). Frans Pretorius in about 2006 was the first to solve this problem by using an adaptation of the hamonic coordinate condition $\partial_i \sqrt{g} g^{ij} = 0.$. One can also put functions of g and pi on the right hand side of this equation without altering the fact that the coordinate changes would travel at the velocity of light. He found a set of such coordinate conditions which gave stable evolution.

Handling the horizons was an equally difficult problem. The problem is that horizons hide singularities, and computers handle singularities very very badly (crash). But inside a black hole, singluarities develop very rapidly. Thus inside a solar mass black hole, the free fall time from the horizon to the singularity is less than 10^{-5} sec. But outside the black hole, the orbital period and the propagation of gravity waves is much slower. One can put a coordinate condition which stops the evolution inside the hole (makes N, the lapse function, go to 0 there) but this stretches the coordinate. making some of the metric coefficients far larger than others– which makes it very hard to get accuracy. However, if one can use a causal evolution, then we know that nothing can get of the horizon. One can therefor put on boundary conditions at the horizon that everything flows into the black hole horizon and one can throw away the solution inside the horizon.

Once I pointed this out, horizons ceased to be a problem. Other techniques (moving puncture methods for example) put on coordinate conditions inside the horizons which make the number of points there very few, and allowed one to handle the singularity as if it were a point source. Again because of causality the evolution inside the horizon did not really matter.

So now one can follow the evolution of for example two blackholes orbiting eachother for as long as desired. the insabilities have been tamed.

0.2 Electromagnetic

There is a lot of similarity between the gravitational initial value problem and the electromagnetic. The electromagnetic Action is

$$I = \frac{1}{4}\int (\partial_j A_i - \partial_i A_j)\eta^{ik}\eta^{jl}(\partial_l A_k - \partial_k A_l) d^3x dt \tag{33}$$

separating out the variables as $A_0 = -\phi,\ A_a = \vec{A}$, these equations can be written as

$$I = \frac{1}{4}\int (-\nabla\phi - \partial_t\vec{A})\cdot(-\nabla\phi - \partial_t\vec{A}) - (\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\vec{A}) d^3x dt \tag{34}$$

Note that there is no time derivative of ϕ in this equation. Thus ϕ has no conjugate momentum. There is however a time derivative of $\vec{A}$, giving as the momentum$\Pi^a = \frac{1}{2}\nabla\phi + \partial_t\vec{A}$. with the consequent Hamiltonian action

$$I = \int \vec{\Pi}\cdot\partial_t\vec{A} - (\vec{\Pi}\cdot\vec{\Pi}) - \frac{1}{4}(\vec{\nabla}\times\vec{A})\cdot(\vec{\nabla}\times\vec{A}) + \vec{P}i\cdot\nabla\phi \tag{35}$$

Doing an integration by parts on that last terms it becomes

$$-\int \phi\nabla\cdot\vec{\Pi} \tag{36}$$

Since ϕ has no conjugate momentum, its variation gives us

$$\nabla\vec{\Pi} = 0 \tag{37}$$

which is a constraint on the intial data. Ie, one cannot specify the intial data for $\vec{A}$ and $\vec{\Pi}$ arbitrarily. Instead the constraint means that the initial data obeys an equation which must be satisfied. $\vec{\Pi}$ is just 1/2 the electric field.

In this case the Hamiltonian is not just a sum of constraints as it is in GR, so even with the constraint satisfied, the Hamiltonian is non-zero. There is a non trivial energy even if the equations of motion are satisfied.

The constaint is also the generator of "gauge" transformations. Defining the Poisson Bracket between two operators A and B which are function of the canonical variables $p_i,\ q_i$ by

$$A, B = \sum_i \partial_{q_i} A\partial_{p_i} B - \partial_{p_i} A\partial_{q_i} B \tag{38}$$

One can regard A as performing an infinitessimal cannonical transformation on B. Thus

$$\delta q_j = q_j, A = \sum_i (\partial_{p_i} A\delta_{ij}) = \partial_{p_j} A\delta p_j = -\partial_{q_j} A \tag{39}$$

Under this canonical transformation, B becomes

$$\delta B = B, A \tag{40}$$

Now let us look at the canonical variable

$$C = \int \psi(\tilde{x})\nabla\cdot\vec{\Pi}(\tilde{x})d^3\tilde{x} \tag{41}$$

$$= -\int \nabla\psi(\tilde{x})\cdot\vec{\Pi}(\tilde{x})d^3\tilde{x} \tag{42}$$

where the last line is by doing an integration by parts and assuming that $\psi(\tilde{x})$ goes to 0 on the boundaries.

It will induce a canonical transformation on $\vec{\Pi}$, $\vec{A}$ of

$$\delta A_j(x) = A_j(x), C = \int \partial_{A_i}(x') A_j(x) \sum_i \partial_{\Pi_i(x')} C d^3x' \tag{43}$$

$$= \int \sum_i \delta(x, x') \delta_i ij \int \nabla_i \psi(\tilde{x}) \delta(x', \tilde{x}) d^3\tilde{x} \tag{44}$$

$$= -\partial_{x_i} \Psi(x) \tag{45}$$

$$\tag{46}$$

and since C is independent of $\vec{A}$, we have

$$\delta\vec{\Pi} = \vec{\Pi}(x), C = 0 \tag{47}$$

This is just a gauge transformation on $\vec{A}$, and Π is not altered by that gauge transformation (which is not surprizing since $\vec{\Pi}$ is $\frac{1}{2}\vec{E}$

The constraints thus generate the "gauge" transformations. In the case of gravity, they are closely related to coordinate transformations.The Momentum constraints really do generate spatial coordinate transformations. The Hamiltonian constraints however generate something similar to time transformations, but it is only on the partial soltuions ($\pi^{ij} - \partial_t \gamma_{ij} + ...$ are they really temporal coordinate transformations. The full Hamiltonian is thus the sum of spatial transformations plus time translations.

Or to put it another way, only on solutions to the equations of motion can the metric at various times be fit together into a 4 metric.

In the EM case,the equivalent to the Harmonic gauge is the Lorentz gfauge, with

$$\partial \mu A^\mu = 0$$

which gives an equation of motion for ϕ. Similarly the Harmonic gauge gives equations of motion for N and for N^i in the gravitational case. The intial data for N and N_a (as for ϕ in EM) are arbitrary. Ie, the Lorentz gauge and the harmonic gauge so not remove all of the gauge freedom.