Einstein's luck?

W. G. Unruh
CIfAR Cosmology and Gravity Program
Dept. of Physics
University of B. C.
Vancouver, Canada V6T 1Z1
Hagler Inst. for Adv Study
Texas A&M Univ.
College Station, Tx email: unruh@physics.ubc.ca

Had Einstein not insisted on the determinant of the metric being the flat spacetime value, he might well have found the wrong value for Mercury's precession

I.

In Nov 1915 Einstein published[**?**] his preliminary theory of gravity (in terms of $R_\mu\nu$ and a few weeks later he published a paper on the advance of the perihelion of Mercury[**?**] in this theory, and his final thoery in terms of what we now call the Einstein tensor. He found a value of 43 seconds of arc per century for this advance, in agreement with the 45 ± 5 seconds of arc per second difference between the the measured value and the predicted value from perturbations by the other planets on Mercury's orbit. This agreement between the prediction of his new theory and observation was an strong argument in favour of his theory.

Unfortunately, Einstein had no exact solution to his field equations. Instead he used the linearized equations, a set of coordinate conditions, and a requirement that the determinant of the metric be equal to 1. His coordinate conditions were that det$g = 1$, $g_{tk} = 0$ (where k goes from 1 to 3).

Unfortunately, his paper is skimpy on details, and he writes his version of the linearized metric by fiat, stating that it obeys the linearized equation, but giving no derivation, nor does he write his equations of motion. Instead he "assumes" a solution. Fortunately for him he chose a very particular solution of the linearized equations, one which corresponds to the linearization of the Schwarzschild solution. There are an infinite set of other solutions, however, almost all of which lead to different values for the perihelion shift.

Let us write the metric in diagonal form in polar coordinates, with the proviso that it be spherically symmetric.

$$ds^2 = \mu_t dt^2 + 2\mu_d dtdr - \mu_r dr^2 - \mu_\phi r^2(d\theta^2 + sin^2(\theta)^2 d\phi^2) \tag{1}$$

where $\mu_t,\ \mu_r,\ \mu_\phi$ are functions only of r.

These can be written in cartesian spatial coordinates as

$$ds^2 = \mu_t dt^2 + 2\mu_d dt\frac{(\vec{x}\cdot d\vec{x}}{r} - (\mu_r - \mu_\phi)\frac{(\vec{x}\cdot d\vec{x})^2}{r^2} - \mu_\phi(dx^2 + dy^2 + dz^2) \tag{2}$$

where $r^2 = \vec{x}\cdot\vec{x}$. In the linearized theory, the demand that that the geodesic equations for slow motion of an object be equivalent to Newton's gravitational equations led to the linearised solution for μ_t of $\mu_{Lt} = 1 - 2M/r$. The other terms need to be determined from the field equations.

The condition he places on the solutions is that they obey det$g = -1$. He saw this as either a coordinate condition or a restriction on the solutions. Part of the problem was that he was on the way to altering his field equation as using the now-called Einstein tensor $G_{\mu\mu}$ rather than $R_{\mu\nu}$, which he presented a week later. This also led to his claim that the Energy momentum tensor must be trace free, a lousy condition for the sun, where, even in the center of sun, the pressures are less than a thousandth of the restmass-energy density. He justified this condition by an appeal to the speculation that the matter is all electromagnetic, and electromagnetism has an energy momentum tensor with zero trace.

In the linearized metric above, the μ_d off diagonal term was abandoned fiat as one of his coordinate conditions. Even 8 years later, when the discussions with Paine-Levi about his solution occured, who felt that his solution was a disproof of Einstein's theory, as one had more than one solution for a spherically symmetric source, Einstein emplasised that coordinate transformations of a solution are equivalent to the solution. Though recognizing their solutions as valid, he seems to ignored Paine-Levi and Gullstrand's identical solutions because they had off diagonal terms in time. That he would have also rejected such a solution for his linearised theory in 1915 is not surprizing. It is unfortunate, since, if one chooses

$$\mu_{Lt} = 1 - \frac{2M}{r} \tag{3}$$

$$\mu_{Ld} = \pm\frac{2M}{r} \tag{4}$$

$$\mu_{Lr} = \frac{1+2M}{r\mu_{L\phi} = 1} \tag{5}$$

which is a solution of the linearized metric, one would have an exact solution for his field equations ($R_{\mu\nu} = G_{\mu\nu} = 0$. The linearized solution is also an exact solution. This metric obeys $\det g = -1$ but clearly not that $g_{tk} = 0$.

Thus, given his apparent prejudice against off-diagonal metrics, let us retain only μ_r and μ_ϕ as unknown functions, with $\mu_{Lt} = 1 - \frac{2M}{r}$. Let us assume that these are functions of only $\frac{M}{r}$. with $\mu_{Lr} = 1 + \alpha\frac{M}{r}$ and $mu_{L\phi} = 1 + \beta\frac{M}{r}$. The condition that the determinant of the metric be -1 gives to first order in $\frac{M}{r}$ that $2 - \alpha - 2\beta = 0$. In the usual way, the geodesic equations give

$$(\frac{du}{d\phi})^2 = -\frac{\mu_\phi^2}{l^2\mu_t\mu_r}(E^2 - \mu_t) - \frac{\mu_\phi}{\mu_r}u^2 \tag{6}$$

where $\frac{u=1}{r}$, $E = \mu_t\frac{dt}{ds}$ and $l = r^2\mu_\phi r^2\frac{d\phi}{ds}$ with $\theta = \pi/2$.

Now, using $\mu_t\mu_r\mu_\phi^2 = 1$ we can elimnate μ_ϕ to get

$$(\frac{du}{d\phi})^2 = \frac{1}{l^2(\mu_t\mu_r)^2}(E^2 - \mu_t) - \frac{1}{(\mu_r^3\mu_t)^{3/2}}u^2 \tag{7}$$

Linearizing this equation, we get

$$(\frac{du}{d\phi})^2 = \frac{1}{l^2}[(1 + (-4M + 2\alpha)u)(E^2 - 1) + 2Mu] - (1 + (3/2\alpha - 1)Mu)u^2 = V(u) \tag{8}$$

Assuming that the orbit is circular allows us to solve $\frac{dV}{du}(u_0) = V(u_0) = 0$ for l^2 and $E^2 - 1$ in terms of u_0, the inverse radius. Defining u_0 as the point were $\frac{d^u}{d\phi^2 = 0}$, and perturbing $E^2 - 1$ and l very slightly while preserving $\frac{dV}{du}(u_0) = 0$, one gets an elliptical orbit where u_0 is the mean value of u in the orbit and the closure of the orbit is when $\sqrt{\frac{1}{2}\frac{d^2V}{du^2}}\Delta\phi = 2\pi$. To lowest order in Mu_0 this gives

$$\Delta\phi = 2\pi(1 - 3/2(1 - 3\alpha/2)Mu_0) \tag{9}$$

where $\Delta\phi$ is the angle required for u to go from maximum (perihelion) to the next maximum. Einstein chose $\beta = 0$ which gave $\alpha = 2$ but it is clear that one can have any value for the perihelion advance in this first order approximation by an appropriate choice of α.

For the Schwarzschild solution, this also gives the solution to lowest order. However, if one changes $\beta = 0$ by for example doing a coordinate transformation on r (eg, $r \to r + \beta M/2$), one can change β. The linearized metric then can give an arbitrary perihelion advance. This however changes the second order metric, and in particular, the second order (in Mu) value of μ_t and μ_r. In particular, this changes the u dependence of the first terms in $V(u)$ to give second order terms. For example, in isotropic coordinates (which do not obey $\det g = -1$, it is the second order value of μ_t which also contributes to the perihelion advance, cancelling the effect of β in the first order expansion. But to know what the second order terms in the metric are, one needs a way of calculating the second order terms. Not even having Einstein's equations for the first order perturbations, it become impossible to know what the second order calculation would have given him.

Ie, Einstein was lucky in choosing his first order solution ($\beta = 0$) to be exactly the first order approximation to the exact Schwarzschild solution. At the same time he was unlucky in that his apparent prejudice against off-diagonal temporal terms in the metric prevented him from finding an exact solution before Schwarzschild did.

It is not clear to me how he found the solution which he assumed. In the the last section of the paper [**?**] he presents as a coordinate condition that $\frac{\partial g^{\alpha\beta}}{\partial x^\alpha} = 0$. His ansatz does not satisfy this equation.

$$\partial_k g^{kl} = \partial_k(\frac{2M}{r^3}x^k x^l \tag{10}$$

$$= \frac{2M}{r^3}(3x^l + \delta^l_k x^k) - 3x^l\frac{x_k x^k}{r^2}x^l)) = \frac{2M}{r^3}x^l. \tag{11}$$