General Relativitiy
Curvature

Riemann Normal coordinates

Consider a Tangent vector T^A defined at a point. Choose an arbitrary set of basis vectors E_i^A at that point, so that we can write $T^A = t^i E_i^A$. (Those could be the tangent vectors to the coordinate axes in some coordinate system). Now define the geodesic throught that point parameterized so that the tangent vector to the geodesic is T^A. Ie, if the geodesic is γ_T, choose the geodesic so that $\left(\frac{\partial}{\partial\gamma_T}\right)^A = T^A$. Define the coordinate $y^i(\gamma_T(1) = t^i$. Since the geodesics with tangent vectors μT^A for a given T^A are all the same geodesic with parameters scaled by μ, we have that in the y coordinate system, the geodesic is given by $y^i = \mu t^i$. But then the geodesic equation in these coordinates becomes

$$\frac{d^2y^i}{d\mu^2} + \Gamma^i_{jk}\frac{dy^j}{d\mu}\frac{dy^k}{d\mu} = 0 \tag{1}$$

or

$$0 = \Gamma^i_{jk}\frac{dy^j}{d\mu}\frac{dy^k}{d\mu} = \Gamma^i_{jk}t^j t^k \tag{2}$$

for all t^i and thus the Γs are 0.

Curvature

Consider two families of curves filling space, such that each set are derived by Lie dragging one set by means of the other $\gamma(\lambda)$ and $\tilde\gamma(\mu)$. This means that the Lie derivative of one set of tangent vectors with respect to the other is zero.

$$\pounds_{\frac{\partial}{\partial\gamma}}\frac{\partial^A}{\partial\tilde\gamma} = \mathit{0} \tag{3}$$

Now consider

$$D_\lambda D_\mu V^A - D_\mu D_\lambda V^A = \lim_{\mu=\lambda=0}\frac{1}{\mu\lambda}(P_\lambda)(P_\mu V^A(\mu,\lambda) - V^A(0,\lambda)) - (P_\mu V^A(\mu,0) - V^A(0,0)) \tag{4}$$

$$-P_\mu(P_\lambda V^A(\mu\lambda) - V(\mu,0)) - P_\lambda V^A(0,\lambda) - V^A(0,0)) \tag{5}$$

$$= \lim_{\mu=\lambda=0}\frac{1}{\mu\lambda}((P_\lambda P_\mu V^A(\mu,\lambda) - P_\mu P_\lambda V^A(\mu,\lambda)) \tag{6}$$

which is clearly linear in $V^A(0,0)$ in the limit.

Now,

$$D_\lambda D_\mu V^A - D_\mu D_\lambda V^A = \eta^C \xi^D (\nabla_C \nabla_D V^A - \nabla_D \nabla_C V^A) + \pounds_\eta \xi^D \nabla_D V^A \quad (7)$$

Since the last term is zero, we have that $(\nabla_C \nabla_D V^A - \nabla_D \nabla_C V^A)$ is linear in V^A and is thus a tensor in that argument. We can thus write this as

$$(\nabla_C \nabla_D V^A - \nabla_D \nabla_C V^A) = R^A{}_{BCD} V^B \quad (8)$$

$R^A{}_{BCD}$ is the Riemann curvature tensor.

Thus the components are

$$\nabla_i \nabla_j V^k = \partial_i \partial_j V^k + \partial_k (\gamma^i_{jl} V^l) - \Gamma^l_{ij} (\partial_l V^k + \Gamma^k_{jm} V^m) + \Gamma^k_{il} (\partial_j V^l + \Gamma^l_{jm} V^m) \quad (9)$$

Antisymmetrizing over ij and using the symmetery of partial derivatives and the symmetry of the Γ we get

$$R^k_{lij} = \partial_i \Gamma^k_{jl} - \partial_j \Gamma^k_{il} + \Gamma^k_{im} \Gamma^m_{jl} - \Gamma^k_{jm} \Gamma^m_{il} \quad (10)$$

Symmetries

The Riemann tensor has a number of symmetries. Firstly it is clear from the definition that

$$R^A{}_{BCD} = -R^A{}_{BCD} \quad (11)$$

Since symmetries of components are symmetries of the tensor itself, we can go into a normal coordinate system where all the first derivatives of the metric (and thus all the Γs) are zero. Then

$$R_{ijkl} = g_{im} R^i{}_{jkl} = \partial_k \Gamma_{ijl} - \partial_j \Gamma_{ikl} \quad (12)$$

where I used that the derivative of the metric was zero, and defined

$$\Gamma_{ijk} = g_{im} \Gamma^m_{jk} = \frac{1}{2} (\partial_j g_{ik} + \partial_k g_{ij} - \partial_i g_{jk}) \quad (13)$$

This gives

$$R_{ijkl} = \frac{1}{2} (\partial_k \partial_j g_{il} + \partial_l \partial_i g_{kj} - \partial_l \partial_j g_{ik} - \partial_k \partial_i g_{lj}) \quad (14)$$

This clearly also satisfies

$$R_{ijkl} = R_{klij} \tag{15}$$
$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \tag{16}$$

Since $R_{ABCD}V^aW^BU^CX^D = R_{ijkl}V^iW^jU^kX^l$, if the components have some symmetry (eg $(R_{ijkl}V^iW^jU^kX^l = -R_{ijkl}V^iW^jU^lX^k$ in any coordinate system for arbitrary vectors, then so does the tensor.

Bianci Indentities

Consider

$$\begin{aligned}\nabla_A(\nabla_B\nabla_C V^D - \nabla_C\nabla_B V^D) - (\nabla_B\nabla_C\nabla_A V^D - \nabla_C\nabla_B\nabla_A V^D) &= \nabla_A({R^D}_{EBC}V^E) - {R^D}_{ECD}\nabla_A V^E \\ &= \nabla_A({R^D}_{EBC})V^E\end{aligned}$$

Using the third symmetry of R we have

$$0 = \nabla_A({R^D}_{EBC}) + \nabla_B({R^D}_{ECA}) + \nabla_C({R^D}_{EAB}) \tag{19}$$

The left hand side is zero identically since each term cancels with another term. This is the Bianci identities.

If we define the contracted tensor

$$R_{BD} = {R^A}_B AD \tag{20}$$
$$R = g_{BD}R^{BD} \tag{21}$$

and contract the Bianci identities between BD and AE and recalling that $\nabla_A g_{BC} = 0$ we get

$$g^{AE}(\nabla_A({R^D}_{EDC}) + \nabla_D({R^D}_{ECA}) + \nabla_C({R^D}_{EAD}) = \nabla_A {R^A}_C + \nabla_D g^{AE}{R^D}_{ECA} + \nabla_C(g^{AE}{R^D}_{EAD}) \tag{22}$$
$$= \nabla_A {R^A}_C + \nabla_D {R_E}^D{}_{CA}\nabla_C(-{R^D}_{EDA} = 2\nabla_A {R^A}_C - \nabla_C R \tag{23}$$

Defining

$$G_{AB} = R_{AB} - \frac{1}{2}g_{AB}R \tag{24}$$

This becomes

$$2\nabla_A {G^A}_B = 0 \tag{25}$$

Note that another tensor which is useful is the completely trace free curvature. In 4 dimensions

$$C_{ABCD} = R_{ABCD} - \frac{1}{2}(R_{AC}g_{BD} - R_{AD}g_{BC} - R_{BC}g_{AD} + R_{BD}g_{AC})) - \frac{1}{6}R(g_{AC}g_{BD} - g_{AD}gBC) \quad (26)$$

which is trace-free. ($g^{AC}C_{ABCD} = 0$). This is called the Weyl tensor, and also has the property that if $\tilde{g}_{AB} = \Omega^2 g_{AB}$, then the Weyl tensor $\tilde{C}^A_{BCD}$ for the confomally transformed metric $\tilde{g}_{AB}$ is the same as for the original tensor ${C^A}_{BCD}$ defined for g_{AB} . Note that C_{ABCD} is zero for all dimensions less than 4. In three dimensions, R_{ABCD} can be written in terms of R_{AB} and in two dimensions both R_{ABCD} and R_{AB} can be written in terms of R and the metric alone.

Linearized curvature

Let us write in some coordinate system that

$$g_{ij} = \eta_{ij} + h_{ij} \quad (27)$$

where the η_{ij} are assumed to be constants in spacetime, and h_{ij} are assumed all to be small, so we will keep only terms to first order in the various h_{ij}.

Then

$$g^{ij} = \eta^{ij} - \eta^{ik}\eta^{jl}h_{jl} \quad (28)$$

as can be seen by

$$\delta^i_j = g^{ik}g_{kj} = \eta^{ik}\eta_{kj} + \eta^{ik}h_{kj} - \eta^{ik}h_{kl}\eta^{lm}\eta_{mj} + O(h^2) = \eta^{ik}\eta_{kj} = \delta^i_j \quad (29)$$

In the curvature, all of the terms that go like $\Gamma\Gamma$ will be second order in h since ${\Gamma^i}_{jk}$ is written in terms of derivatives of the h and thus is first order in h, and products would be second order.

Also $g_{im}\partial_k\Gamma^m_{jl} = \partial_k\Gamma_i jk + O(h^2)$ and thus the linearized curvature to lowest order in h is the same as the above curvture in Riemann normal coordinates

$$R_{ijkl} = \frac{1}{2}(\partial_k\partial_j h_{il}\partial_l\partial_i h_{jk} - \partial_k\partial_i h_{jl} - \partial_l\partial_j h_{ik}) \quad (30)$$

The Ricci curvature is

$$R_{jk} = \frac{1}{2}(\partial_k\partial_j h + \Box h_{ij} - \partial_k\partial i\eta^{il}h_{lj} - \partial_j\partial_i\eta^{il}h_{lk}) \quad (31)$$

where $h = \eta^{ij} h_{ij}$ and $\Box = \eta^{ij}\partial_i\partial_j$

If we write $\bar{h}_{ij} = h_{ij} - \frac{1}{2}h\eta_{ij}$, then we have

$$G_{ij} = \frac{1}{2}(\Box\bar{h}_{ij} - \partial_i\partial_l(\eta^{lk}\bar{h}_{kj} - \partial_j\partial_l\eta^{lk}\bar{h}_{li}) \tag{32}$$

If η_{ij} is the Minkowski metric, then $\Box$ is like a wave equation, and the other two terms are divergences. Since the small metric cchanges if one performs coordinate transformations, this gives us hope that perhaps those divergences can be set to zero, and the then G_{ij} is just a wave equation. (This is similar to electromagmetism, where the equation for A^ii, the vector potential, is of the form

$$\Box A^i - \eta^{ij}\partial_j\partial_k A^k = J^i \tag{33}$$

and the second term can be eliminated via a guage transformation.