General Relativitiy
Field theory

Quantum Field Theory

The application of the previous theory to field theory is straightforward. Instead of the index i, one takes the index to be the spatial point x and teh sum converts to an integral over x. The Lagrangian, in flat spacetime is

$$L = \frac{1}{2}\int(\partial_t\phi(t,x))^2 - (\nabla\phi\cdot\nabla\phi) - m^2\phi)d^3x \tag{1}$$

with the momentum being

$$\pi(t,x) = \partial_t\phi \tag{2}$$

and the Hamiltonian

$$H = \frac{1}{2}\int(\pi^2 + (\nabla\phi\cdot\nabla\phi) + m^2\phi^2)d^3x \tag{3}$$

The inner product between two complex solutions to the equations are

$$< \tilde{\phi}, \phi >= i\int(\tilde{\pi}(t,x)^*\phi(t,x) - \tilde{\phi}(t,x)^*\pi(t,x))d^3x \tag{4}$$

$$= i\int(\dot{\tilde{\phi}(t,x)}^*\phi(t,x) - \tilde{\phi}(t,x)^*\phi(t,x))d^3x \tag{5}$$

One can choose some arbitrary set of complex solutions, indexed by α namely $\phi_\alpha(t,x)$ which we demand that

$$< \phi_\alpha, \phi_\beta >= \delta_{\alpha\beta} \tag{6}$$

$$< \phi^*_\alpha, \phi_\beta >= 0 \tag{7}$$

Then we can define the annihilation operators

$$A_\alpha =< \phi_\alpha, \Phi > \tag{8}$$

where Π, Φ are the quantum operators which obey the field equations. Then we can write

$$\Phi = \sum_\alpha(A_\alpha\phi_\alpha(t,x) - A^\dagger_\alpha\phi^*(t,x)) \tag{9}$$

The Hamiltonian diagonalization is given by

$$\partial_t \phi_\omega = i\omega\phi_\omega \tag{10}$$

Choose the modes

$$\phi(t,x) = \phi_k \frac{e^{-ikx}}{\sqrt{(2\pi)^3}} \tag{11}$$

and similarly for π. The equations of motion are

$$\dot{\phi}_k = \pi_k \tag{12}$$

$$\pi_k = -(k^2+m^2)\phi_k \tag{13}$$

which gives

$$\omega = \sqrt{k^2+m^2} \tag{14}$$

The annihilation operator

$$A_{\omega k} = i\int \pi_k^* \Phi(t,x) - \phi_k^* \Pi(t,x) \frac{e^{ikx}}{\sqrt{(2\pi)^3}} d^3x \tag{15}$$

The annihilation operators $A_{\omega,k}$ also minimize the energy, and the state annihilated $A_{\omega k}|0\rangle$ is the usual vacuum.

Let us now choose a more complex situation. Write the Lagrangian as

$$L = \frac{1}{2}\int a(t)^3((\partial_t\phi)^2 - \frac{1}{a(t)^2}(\nabla\phi)^2 - m^2\phi^2)d^3x \tag{16}$$

This is just

$$\frac{1}{2}\int \sqrt{|g|}g^{\mu\nu}(\partial_\mu\phi\partial_\nu\phi)d^3x \tag{17}$$

the coordinate invariant Lagrangian for the scalar field in the cosmological metric

$$ds^2 = dt^2 - a(t)^2(dx^2+dy^2+dz^2) \tag{18}$$

The conjugate momentum will be

$$\pi = a^3\dot{\phi} \tag{19}$$

to give a Hamiltonian

$$H = \frac{1}{2}\int (\frac{\pi^2}{a^3} + a(\nabla\phi)^2 + a^3 m^2\phi^2) d^3x \tag{20}$$

The inner product is again

$$< \phi, \tilde{\phi} >= i\int (\pi^*\tilde{\phi} - \phi^*\tilde{\pi}) d^3x \tag{21}$$

Choose the modes

$$\phi = \phi_k \frac{e^{-ikx}}{\sqrt{(2\pi)^3}} \tag{22}$$

and similarly for π. The Hamiltonian diagonalization at time t gives the eigenvectors and eigenvalues

$$i\omega\pi_{\omega k} = -(ak^2 + a^3m^2)\phi_{\omega k} \tag{23}$$

$$i\omega\phi_{\omega k} = \frac{\pi_{\omega k}}{a^3} \tag{24}$$

$$\omega = \sqrt{\frac{k^2}{a^2} + m^2} \tag{25}$$

ω depends on time. These modes are not solutions of the equations of motion unless a is independent of time. However we can use the modes at the time t at which they are defined as above as inital data for a complete solution. In order to normalise the mode, we want

$$\delta(k, k') = i\int (\pi_k^*\phi_k' - \phi_k^*\pi_k') \int \frac{e^{i(k-k')x}}{(2\pi)^3} d^3x \tag{26}$$

$$= 2\omega a^3|\phi_k|^2\delta(k - k') \tag{27}$$

or

$$\phi_k = \frac{1}{\sqrt{2\omega a^3}} \tag{28}$$

The equations of motion in the Heisenberg representation can be written for $\Phi_k(t) = \int \frac{1}{\sqrt{(2\pi)^3}} e^{ikx}\Phi(t,x)$ and similarly for $\Pi_k(t)$. At time $t = t_0$ we have the equation

$$\dot{\Phi}_k(t) = \frac{1}{a(t_0)^3} Pi_k(t_0) \tag{29}$$

$$\dot{\Pi}_k(t) = -(a(t_0)k^2 + a^3(t_0))\Phi_k(t_0) = -\omega^2 a^3(t)\Phi_k \tag{30}$$

Thus we can define the Annihilation operators corresponding to this Hamiltonian diagonalisation at each time t.

$$A_k(t) < \phi_k e^{-ikx}, \Phi >= \sqrt{\frac{a^3\omega}{2}}\Phi_k - \frac{i}{\sqrt{2\omega a^3}}\Pi_k \tag{31}$$

where

$$\Phi_k = \int \Phi \frac{e^{ikx}}{\sqrt{(2\pi)^3}} d^3x \tag{32}$$

and similarly for Π_k.

Then

$$\frac{dA_k}{dt} = \sqrt{\frac{a^3\omega}{2}}\dot{\Phi}_k - \frac{i}{\sqrt{2\omega a^3}}\dot{\Pi}_k + \partial_t \ln(\sqrt{a^3\omega})(A^\dagger) \tag{33}$$

$$= i\omega A_k + \frac{\dot{a}}{a}(3 - \frac{k^2}{k^2 + m^2 a^2})A_k^\dagger \tag{34}$$

and the Hamiltonian diagonaisation annihilation operator at time $t + \delta t$ is a mixture of the annihilation operators at time t. If one starts in the vacuum state at time t or its annihilation operators, it is not the vacuum state at time $t + \delta t$ but is rather a squeezed state. Defining the the number of particles at time $t + \delta t$ by the operator $N(t + \delta t) = A_k^\dagger(t + \delta t)A_k(t_\delta t)$ where the state is the vacuum state with respect to $A_k(t)|0_t\rangle = 0$

$$\langle 0_t|N(t + \delta t)|0_t\rangle \approx \delta t^2 \left(\frac{\dot{a}}{a}(3 - \frac{k^2}{k^2 + m^2 a^2})\right)^2 \tag{35}$$

As $k \to \infty$, this approaches a constant. Thus when integrated over all modes k this diverges as $k^3\delta t^2$. If one defines ones particles by Hamiltonian diagonalization, then after the smallest instant of time, a vacuum state, a no-particle state, is converted into one with and infinite number of particles in it. (and an infinite energy compared to the minimum energy).

This problem was recognized by Parker in the late 60's to early 70's. It led, and still leads, to huge controversies as to what is meant by particles or excitations in such a cosmological spacetime.

One approach is to change the definition of time. If we define a new time with

$$\tau = \int \frac{dt}{a(t)} \tag{36}$$

the so called conformal time, the Hamiltonian becomes

$$H_\tau = \frac{\Pi^2}{a^2} + a^2(\nabla\phi)^2 + m^2 a^3 \Phi^2. \tag{37}$$

Now, let $\hat{\Phi} = a\Phi$. We can define the new conjugate momentum as

$$\hat{\Pi} = \frac{\Pi}{a} + \partial_t(a)\Phi \tag{38}$$

The new Hamiltonian, which gives the same equations of motion for Φ as the old Hamiltonian did becomes

$$\hat{H} = \frac{1}{2}(\hat{\Pi}^2 + (\nabla\hat{\Phi})^2 + (m^2a^2 - \frac{\partial_\tau^2 a}{a} - (\frac{\partial_\tau^2 a}{a})^2)\hat{\Phi}^2 \tag{39}$$

A direct calculation shows that the equations of motion given by this Hamiltonian in terms of the time τ are the same as before. Defining $h = \frac{\partial_\tau a}{a}$, we can find the Hamiltonian diagonalisation for this Hamiltonian from which we find

$$\hat{\omega} = \sqrt{k^2 + m^2a^2 - h^2 - \partial_\tau h} \tag{40}$$

and the normalisation factor for the solutions are

$$\phi_k = \frac{1}{\sqrt{2\hat{\omega}}} \tag{41}$$

which gives

$$\hat{A}_k(\tau) = \frac{1}{\sqrt{2}}(\sqrt{\hat{\omega}}\hat{\Phi}_k - \frac{i}{\sqrt{\hat{\omega}}}\hat{\Pi}_k) \tag{42}$$

and

$$\partial_\tau A_k = i\hat{\omega} A_k + \frac{\partial_\tau \hat{\omega}^2}{4\hat{\omega}^2} A_k^\dagger \tag{43}$$

The number of particles after time $\delta\tau$ then becomes

$$\langle \hat{0}_\tau | \hat{A}_k^\dagger(\tau + \delta\tau) \hat{A}_k(\tau + \delta\tau) | \hat{0}_\tau \rangle \approx O\left(\frac{\delta\tau^2}{k^4}\right) \tag{44}$$

and the total number of particles which thus goes as $\frac{\int k^2 dk}{k^4}$ for large k is finite as long as m is large enough . Ie, by simply redefining the time, and by redefining the field strength and the conjugate momentum, one can turn an infinite number of particles into a finite number. Does this mean that this finite number is the true particle creation rate? The answer is again no, because this transformation really has no physics behind it. It is an arbitrary mathematical manipulation. Note also that if m is small enough and the hubble factor approriate, $\hat{\omega}$ can become imaginary for small k and small enough t. One could make further redefinitions of the momentum and configuration variables, dependent for example on k, so as to make the convergence at large k even faster. Thus the particle number becomes a somewhat meaningless concept.

0.1 Appendix

/bf Changing variables in Hamiltonian

The Hamiltonian action is

$$S = \int p_i (\sum_i \partial_t q_i - H(p,q)) dt \tag{45}$$

To change configuration or momentum variables, we must preserve the form of this action. Thus, for example if we change q_i to $\tilde{q}_i = \alpha q_i$ we must also change p_i so $\hat{p}^i = \frac{1}{\alpha} p_i$. If α depends on time, then we have

$$(p_i \partial_t q_i) dt = \alpha \hat{p}_i \partial_t (\frac{1}{\alpha} q_i) = \hat{p}_i \partial_t \hat{q}_i - \partial_t (\ln(\alpha)) \hat{p}_i \hat{q}_i \tag{46}$$

The last term is part of the action, and is a function of just $\hat{p}_i$ and $\hat{q}_i$ and can therefore be incorporated into H. Thus we have

$$\hat{H}(\hat{p},\hat{q}) = H(\alpha\hat{p}, \frac{q}{\alpha}) + \partial_t(\ln(\alpha))\hat{p}_i\hat{q}_i \tag{47}$$

Another transformation one can make is to let

$$\tilde{p}_i = p_i + \beta q_i \tag{48}$$

$$\tilde{q}_i = q_i \tag{49}$$

and we get

$$p_i\partial_t q_i = (\tilde{p}_i - \beta q_i)\partial_t\tilde{q}_i = \tilde{p}_i\partial_t\tilde{q}_i + \frac{1}{2}(\partial_t\beta)q_i^2 - \partial_t(\beta q_i^2)) \tag{50}$$

The first term is just the usual term in the Hamiltonian action. The second term is one that should be incorporated into the Hamiltonian, and the third term is a complete derivative, and results in boundary terms in the action integral. If the variations of the action are to be taken so that δq is zero on the boundaries (necessary to get the usual Hamilton equations), then this third term will contribute nothing to the variation and can be eliminated. Thuys we have

$$\tilde{H}(\tilde{p},\tilde{q}) = H(\tilde{p} - \beta\tilde{q}, q) - \frac{1}{2}\partial_t\beta q_i^2) \tag{51}$$

For a single degree of freedom, one can continue in this fashion, using the second type of transformation to elimate cross terms (pq) in the Hamiltonian, (but to introduce more comples configuration dependent potentials) and the second to get rid of dependence in the Mass terms or the potential terms, at the expense of introducing cross terms into the Hamiltonian.