PROCEEDINGS OF THE ROYAL SOCIETY A | MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

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Proc. R. Soc. Lond. A 1951 **209**, doi: 10.1098/rspa.1951.0200, published 23 October 1951

Again, it may be remarked that, in this case, such a solution around O implies that, at $t \to \infty$, the ultimate steady flow round the corner through a straight shock is built up by the uniform expansion in time of the steady flow region OYK. On the other hand, when it is not possible to have an attached shock through O, there can be no ultimate steady flow with an infinite wedge.

Acknowledgement is made to the Chief Scientist, Ministry of Supply, and to the Controller of H.M. Stationery Office for permission to publish this paper.

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Spinning test-particles in general relativity. I

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(*Communicated by R. E. Peierls, F.R.S.—Received* 3 *May* 1951)

A method for the derivation of the equations of motion of test particles in a given gravitational field is developed. The equations of motion of spinning test particles are derived. The transformation properties are discussed and the equations of motion are written in a covariant form.

1. Introduction

The problem of the equations of motion in general relativity demands for its solution the determination, at the same time, of the gravitational field of the moving bodies. If one remembers how difficult it is to determine the gravitational field of a given distribution of matter—even in the simplest case of one body at rest—one sees immediately that the problem of the equations of motion will be one of extreme complexity when considered in its general form.

An important simplification occurs when one of the moving bodies is very small compared with the other bodies, so small that its influence on the motion of the other bodies can be neglected. Such a small body is usually called a *test particle*. The problem of the motion of test particles is simpler than the general one because it splits into two separate problems. First, one has to determine the basic gravitational field which will be there when the presence of the test particle is ignored. Then one has to find the equations of motion of the test particle in this basic field, which now is assumed to be already known. This second problem—which, in a restricted sense, may be called the problem of the motion of test particles—is the subject of the present paper.

Historically, what was discussed first was the problem of the motion of bodies with comparable masses, under the restrictive assumption that their velocities are small compared with that of light. The first solution of this problem was given by Einstein and co-workers (1938, 1940, 1949). The method used by those authors has two characteristic features: use is made of the field equations only, and any consideration of the interior of the bodies is avoided. A second method has been proposed by Fock (1939) and further developed by the present author (1951). Its main features are, first, that the interiors of the bodies are also considered, and secondly, that besides the field equations use is made of the 'dynamical equation'

$$\mathfrak{T}^{\mu\nu}{}_{;\nu} = 0.$$

This second method is essentially simpler than the first. It turns out that in order to determine the equations of motion to a given approximation the second method requires the calculation of the quantities $g_{\mu\nu}$ to an approximation by one step lower than the first method.

The motion of test particles has been discussed for the first time by Infeld & Schild (1949). The method used by those authors is that of Einstein and co-workers. They are considering the simplest kind of a test particle which we might call the *single-pole particle*. Their result is that the orbits of such particles are the geodesics of the basic field $g_{\mu\nu}$.

In the present paper the motion of test particles will be discussed from the point of view of the method of Fock. The results are more striking than with the slowly moving bodies of comparable masses. It is found that when the basic field is known, the motion of test particles is determined by the dynamical equation only. The single-pole particle will be treated here as an illustrative example; the main problem will be the motion of test particles whose internal structure requires for their description not only single-pole but also dipole terms, briefly the motion of *pole-dipole particles*. The method can also be used for particles with higher multipoles, so that it can be said that it solves the problem of the motion of test particles in its general form.

The very use of the word dipole in the theory of gravitation will cause distrust among many physicists. The reason is that such a dipole immediately suggests a mass density assuming positive and negative values, a case which does not occur in macroscopic physics. However, this argument has a meaning only so long as we consider the Newtonian theory of gravitation. In general relativity all ten components $\mathfrak{T}^{\mu\nu}$ will be of importance. Now a rotating particle has, necessarily, non-vanishing moments of the components $\mathfrak{T}^{i4}$ ($i = 1, 2, 3$).* It is concluded, therefore, that in general relativity a spinning particle is a pole-dipole particle. This result will be sufficient to show that the motion of pole-dipole particles is not at all a question of a purely mathematical character. On the contrary, it constitutes an important physical problem.

It will be useful to say a word about the accuracy of the equations of motion which will be derived. If it is assumed that the mass and size of the test particle tend to zero, the equations of motion will be rigorous in all respects but one, namely, the neglect

* Throughout this paper Latin indices will take the values 1, 2, 3 only, while Greek indices will take the values 1, 2, 3 and 4.

of the gravitational radiation emitted by the test particle in its motion. The situation is analogous to that of the motion of a charged test particle in a given electromagnetic field, when the electromagnetic radiation emitted by the particle is disregarded. But in view of the extreme smallness of the gravitational radiation it may be expected that the equations of motion will be more than adequate for most physical problems.

2. Description of the method: application to the single-pole particle

In this section the method will be described in detail, and then applied to derive the equations of motion of a single-pole particle.

It is assumed that the dimensions of the test particle are very small compared with the characteristic length of the basic field (e.g. the distance from the central body in the case of the Schwarzschild field). In order to attach a rigorous meaning to the calculation it should be demanded that the dimensions of the particle tend to zero. Thus the particle will describe a narrow tube in the four-dimensional space. Inside this tube a line L is chosen which will 'represent' the motion of the particle. The co-ordinates of the points of L will be denoted by X^α; they may be considered as functions either of $t\,(X^4 = t)$ or of the proper time s on L. The demand that the particle be very small can be formulated in the following way. The tensor $\mathfrak{T}^{\mu\nu}$ describing the particle must be different from zero only within a sphere whose centre is the point with co-ordinates X^i and the radius a very small length R (this being true for any t). When the limiting case $R \to 0$ is assumed, the arbitrariness in the choice of L disappears.

Put
$$\delta x^\alpha = x^\alpha - X^\alpha, \tag{2·1}$$
and consider integrals of the form
$$\int \mathfrak{T}^{\mu\nu}\,dv, \quad \int \delta x^\rho \mathfrak{T}^{\mu\nu}\,dv, \quad \int \delta x^\rho\,\delta x^\sigma\,\mathfrak{T}^{\mu\nu}\,dv, \quad \ldots,$$
the integration being carried over the three-dimensional space for $t = \text{const.}$ The structure of a test particle can now be defined quantitatively. A *single-pole* particle is one which has at least some of the integrals $\int \mathfrak{T}^{\mu\nu}\,dv \neq 0$, while all integrals with one or more factors δx^ρ are equal to zero. A *pole-dipole* particle has at least some of the integrals $\int \mathfrak{T}^{\mu\nu}\,dv$ and $\int \delta x^\rho \mathfrak{T}^{\mu\nu}\,dv \neq 0$, while all integrals with more than one factor δx^ρ are equal to zero. In the same way particles with higher multipoles can be defined.*

* In practical cases these results will be used as successive steps of an approximation procedure. For example, consider a small rotating planet moving around the sun. To the first approximation it will be treated as a single-pole particle. In the next step account is taken of the spin, treating the planet as a pole-dipole particle; the arbitrariness in the choice of the line L may, in this case, be avoided by picking up the centre of mass of the planet, in which case the components S^{i4} of the spin-tensor will vanish. There are also higher multipole terms which can be considered in a similar way. The justification of this procedure lies in the fact that the importance of higher multipoles is decreasing rapidly when the particle is very small; this is so because the additional terms corresponding to a multipole of order n will contain the factor $(R/r)^n$.

The equations of motion of a single-pole particle are now derived. The dynamical equation is written in the form

$$\frac{\partial \mathfrak{T}^{\alpha\beta}}{\partial x^{\beta}} + \Gamma^{\alpha}_{\mu\nu} \mathfrak{T}^{\mu\nu} = 0. \tag{2·2}$$

It follows from (2·2) that

$$\frac{\partial}{\partial x^{\gamma}} (x^{\alpha} \mathfrak{T}^{\beta\gamma}) = \mathfrak{T}^{\beta\alpha} - x^{\alpha} \Gamma^{\beta}_{\mu\nu} \mathfrak{T}^{\mu\nu}. \tag{2·3}$$

Integrating (2·2) and (2·3) over the three-dimensional space for t = const., then

$$\frac{d}{dt}\int \mathfrak{T}^{\alpha 4} dv = -\int \Gamma^{\alpha}_{\mu\nu} \mathfrak{T}^{\mu\nu} dv, \tag{2·4}$$

$$\frac{d}{dt}\int x^{\alpha} \mathfrak{T}^{\beta 4} dv = \int \mathfrak{T}^{\alpha\beta} dv - \int x^{\alpha} \Gamma^{\beta}_{\mu\nu} \mathfrak{T}^{\mu\nu} dv. \tag{2·5}$$

Inside the test particle $\Gamma^{\alpha}_{\mu\nu}$ can be developed in power series:

$$\Gamma^{\alpha}_{\mu\nu} = {}_0\Gamma^{\alpha}_{\mu\nu} + {}_0\Gamma^{\alpha}_{\mu\nu,\sigma} \delta x^{\sigma} + \dots, \tag{2·6}$$

the subscript 0 meaning values at the point X^{α}. Introduce (2·6) in (2·4) and (2·5), and keep in mind that the particle is a single-pole. Then, omitting the subscript 0 which is no longer necessary,

$$\frac{d}{dt}\int \mathfrak{T}^{\alpha 4} dv + \Gamma^{\alpha}_{\mu\nu} \int \mathfrak{T}^{\mu\nu} dv = 0, \tag{2·4a}$$

$$\int \mathfrak{T}^{\alpha\beta} dv = \frac{dX^{\alpha}}{dt} \int \mathfrak{T}^{\beta 4} dv. \tag{2·5a}$$

Now introduce the quantities $M^{\alpha\beta}$ by the relation

$$M^{\alpha\beta} = u^4 \int \mathfrak{T}^{\alpha\beta} dv, \tag{2·7}$$

where $u^4 = \dfrac{dX^4}{ds}$, $ds^2 = g_{\mu\nu} dX^{\mu} dX^{\nu}$. Equations (2·4$a$) and (2·5$a$) take, then, the form

$$\frac{d}{ds}\left(\frac{M^{\alpha 4}}{u^4}\right) + \Gamma^{\alpha}_{\mu\nu} M^{\mu\nu} = 0, \tag{2·8}$$

$$M^{\alpha\beta} = u^{\alpha} \frac{M^{\beta 4}}{u^4}, \tag{2·9}$$

where $u^{\alpha} = \dfrac{dX^{\alpha}}{ds}$. Putting $\beta = 4$ in (2·9) then

$$M^{\alpha 4} = u^{\alpha} \frac{M^{44}}{u^4},$$

and introducing this in the right-hand side of (2·9)

$$M^{\alpha\beta} = m u^{\alpha} u^{\beta}, \tag{2·9a}$$

where $m = M^{44}/(u^4)^2$. With (2·9a) one gets from (2·8)

$$\frac{d}{ds}(m u^{\alpha}) + \Gamma^{\alpha}_{\mu\nu} m u^{\mu} u^{\nu} = 0. \tag{2·10}$$

This is the equation of motion of the single-pole particle. It contains not only the differential equation of the orbit, but also an equation for the quantity m. This last equation can be separated if (2·10) is multiplied by u_α. Taking into account the general relation

$$u_\alpha \frac{du^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} u_\alpha u^\mu u^\nu = 0$$

then

$$\frac{dm}{ds} = 0. \tag{2·11}$$

With this result equation (2·10) reduces to

$$\frac{du^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu = 0, \tag{2·12}$$

which is identical with the result of Infeld & Schild: The orbits of a single-pole particle are the geodesics of the field $g_{\mu\nu}$.

The quantity m is the rest-mass of the test particle. From (2·11) it is seen that the rest-mass of a single-pole particle remains constant throughout the motion. Further, it is possible to prove by a direct calculation that the quantity m is a *scalar* for general co-ordinate transformations. The proof is not given here because the same result will follow from the more general invariance properties which will be proved for the pole-dipole particle (§ 4).

3. The pole-dipole particle

The equations of motion of the pole-dipole particle may now be derived. Besides (2·2) and (2·3) the equation

$$\frac{\partial}{\partial x^\delta}(x^\alpha x^\beta \mathfrak{T}^{\gamma\delta}) = x^\beta \mathfrak{T}^{\gamma\alpha} + x^\alpha \mathfrak{T}^{\gamma\beta} - x^\alpha x^\beta \Gamma^\gamma_{\mu\nu} \mathfrak{T}^{\mu\nu} \tag{3·1}$$

must be used, which is again a direct consequence of (2·2). Now integrate equations (2·2), (2·3) and (3·1) over the three-dimensional space for $t = \text{const.}$, considering (2·6) and (2·1) and remembering that the test particle is now a pole-dipole particle (i.e. one for which all integrals $\int \delta x^\rho \ldots \mathfrak{T}^{\mu\nu} dv$ with more than one factor δx^ρ vanish). Writing $\Gamma^\alpha_{\mu\nu}$ and $\Gamma^\alpha_{\mu\nu,\sigma}$ instead of ${}_0\Gamma^\alpha_{\mu\nu}$ and ${}_0\Gamma^\alpha_{\mu\nu,\sigma}$, then from (2·2)

$$\frac{d}{dt}\int \mathfrak{T}^{\alpha 4} dv + \Gamma^\alpha_{\mu\nu}\int \mathfrak{T}^{\mu\nu} dv + \Gamma^\alpha_{\mu\nu,\sigma}\int \delta x^\sigma \mathfrak{T}^{\mu\nu} dv = 0. \tag{3·2}$$

Then from (2·2), taking into account (3·2),

$$\int \mathfrak{T}^{\alpha\beta} dv = \frac{dX^\alpha}{dt}\int \mathfrak{T}^{\beta 4} dv + \frac{d}{dt}\int \delta x^\alpha \mathfrak{T}^{\beta 4} dv + \Gamma^\beta_{\mu\nu}\int \delta x^\alpha \mathfrak{T}^{\mu\nu} dv. \tag{3·3}$$

And finally from (3·1), taking into account (3·2) and (3·3),

$$\frac{dX^\alpha}{dt}\int \delta x^\beta \mathfrak{T}^{\gamma 4} dv + \frac{dX^\beta}{dt}\int \delta x^\alpha \mathfrak{T}^{\gamma 4} dv = \int \delta x^\alpha \mathfrak{T}^{\beta\gamma} dv + \int \delta x^\beta \mathfrak{T}^{\alpha\gamma} dv. \tag{3·4}$$

Equations (3·2) to (3·4) contain the equations of motion of the pole-dipole particle, though in a disguised form and mixed with other relations. The following calculations will be necessary for bringing the equations of motion into their usual form.

Introduce the quantities $M^{\lambda\mu\nu}$ by the relation

$$M^{\lambda\mu\nu} = -u^4 \int \delta x^\lambda \mathfrak{T}^{\mu\nu}\, dv. \tag{3·5}$$

Note that

$$M^{4\mu\nu} = 0, \tag{3·5a}$$

since all integrals refer to the hyperplane $t = \text{const.}$ and therefore $\delta x^4 = 0$. Furthermore, introduce the components of the spin of the particle

$$S^{\alpha\beta} = \int \delta x^\alpha \mathfrak{T}^{\beta 4}\, dv - \int \delta x^\beta \mathfrak{T}^{\alpha 4}\, dv. \tag{3·6}$$

Because of (3·5) one also writes

$$u^4 S^{\alpha\beta} = -(M^{\alpha\beta 4} - M^{\beta\alpha 4}). \tag{3·6a}$$

Equation (3·4) takes, now, the form

$$u^4(M^{\alpha\beta\gamma} + M^{\beta\alpha\gamma}) = u^\alpha M^{\beta\gamma 4} + u^\beta M^{\alpha\gamma 4}. \tag{3·7}$$

This relation enables $M^{\lambda\mu\nu}$ to be expressed in terms of u^α and $S^{\alpha\beta}$. Two other relations of the form (3·7) may be written by taking the cyclic permutations of the indices α, β, γ. Adding the first and third of these relations and subtracting the second, then

$$2u^4 M^{\alpha\beta\gamma} = u^\alpha(M^{\beta\gamma 4} + M^{\gamma\beta 4}) + u^\beta(M^{\alpha\gamma 4} - M^{\gamma\alpha 4}) + u^\gamma(M^{\alpha\beta 4} - M^{\beta\alpha 4}).$$

In this relation the last four terms can be expressed in terms of $S^{\alpha\beta}$ by means of (3·6a). For the transformation of the remaining two terms one writes (3·7) with $\gamma = 4$:

$$u^4(M^{\alpha\beta 4} + M^{\beta\alpha 4}) = u^\alpha M^{\beta 44} + u^\beta M^{\alpha 44}.$$

According to (3·6a) and (3·5a)

$$u^4 S^{\alpha 4} = -M^{\alpha 44}, \tag{3·6b}$$

and

$$M^{\alpha\beta 4} + M^{\beta\alpha 4} = u^\alpha S^{4\beta} + u^\beta S^{4\alpha}.$$

Hence

$$2M^{\alpha\beta\gamma} = -(S^{\alpha\beta}u^\gamma + S^{\alpha\gamma}u^\beta) + \frac{u^\alpha}{u^4}(S^{4\beta}u^\gamma + S^{4\gamma}u^\beta). \tag{3·8}$$

Next consider equation (3·3). With the help of (2·7) and (3·5) equation (3·3) is written in the form

$$M^{\alpha\beta} = u^\alpha \frac{M^{\beta 4}}{u^4} - \frac{d}{ds}\left(\frac{M^{\alpha\beta 4}}{u^4}\right) - \Gamma^\beta_{\mu\nu} M^{\alpha\mu\nu}. \tag{3·9}$$

This relation enables one to express $M^{\alpha\beta}$ in terms of M^{44}, u^α and $S^{\alpha\beta}$. First, it is found that, if $\beta = 4$ is put in (3·9),

$$M^{\alpha 4} = u^\alpha \frac{M^{44}}{u^4} - \frac{d}{ds}\left(\frac{M^{\alpha 44}}{u^4}\right) - \Gamma^4_{\mu\nu} M^{\alpha\mu\nu}, \tag{3·9a}$$

and then, introducing (3·9a) in (3·9),

$$M^{\alpha\beta} = \frac{u^\alpha}{u^4}\left[\frac{u^\beta}{u^4} M^{44} - \frac{d}{ds}\left(\frac{M^{\beta 44}}{u^4}\right) - \Gamma^4_{\mu\nu} M^{\beta\mu\nu}\right] - \frac{d}{ds}\left(\frac{M^{\alpha\beta 4}}{u^4}\right) - \Gamma^\beta_{\mu\nu} M^{\alpha\mu\nu}. \tag{3·10}$$

Besides this result equation (3·9) contains another important relation. It is found immediately if one remembers that, according to the definition (2·7), it is $M^{\alpha\beta} = M^{\beta\alpha}$. Taking into account (3·6a), then

$$\frac{u^\alpha}{u^4} M^{\beta 4} - \frac{u^\beta}{u^4} M^{\alpha 4} + \frac{dS^{\alpha\beta}}{ds} + \Gamma^\alpha_{\mu\nu} M^{\beta\mu\nu} - \Gamma^\beta_{\mu\nu} M^{\alpha\mu\nu} = 0. \tag{3·11}$$

Or, with the help of (3·9a) and (3·6b),

$$\frac{dS^{\alpha\beta}}{ds} + \frac{u^\alpha}{u^4}\frac{dS^{\beta 4}}{ds} - \frac{u^\beta}{u^4}\frac{dS^{\alpha 4}}{ds} + \left(\Gamma^\alpha_{\mu\nu} - \frac{u^\alpha}{u^4}\Gamma^4_{\mu\nu}\right) M^{\beta\mu\nu} - \left(\Gamma^\beta_{\mu\nu} - \frac{u^\beta}{u^4}\Gamma^4_{\mu\nu}\right) M^{\alpha\mu\nu} = 0. \tag{3·12}$$

This relation contains time derivatives of the quantities $S^{\alpha\beta}$ only, i.e. it is what may be called the equation of motion of the spin.

Finally, it is necessary to transform equation (3·2). This can be written

$$\frac{d}{ds}\left(\frac{M^{\alpha 4}}{u^4}\right) + \Gamma^\alpha_{\mu\nu} M^{\mu\nu} - \Gamma^\alpha_{\mu\nu,\sigma} M^{\sigma\mu\nu} = 0, \tag{3·13}$$

with $M^{\alpha 4}$ and $M^{\mu\nu}$ defined by (3·9a) and (3·10). Equation (3·13) is the ordinary equation for the motion of the particle in the space.

The total number of the unknowns entering in these equations is ten: M^{44}, three independent components u^α and six components $S^{\alpha\beta}$. A quick glance at equations (3·12) and (3·13) would seem to show that there are exactly as many equations. This is, however, not the case, because three of equations (3·12)—those corresponding to $\alpha = 1, 2, 3$ and $\beta = 4$—are trivial identities. Hence the number of the equations of motion is *not sufficient* for the determination of all unknowns, i.e. for the complete determination of the motion.

This is nothing really new. It has been noticed already when the general pole-dipole particle has been discussed in special relativity (Mathisson 1937; Lubanski 1937; Hönl & Papapetrou 1939a, b). The results of that discussion can be obtained immediately from (3·11) and (3·13) by putting $\Gamma^\alpha_{\mu\nu} = 0$. One finds from (3·13)

$$\frac{dP^\alpha}{ds} = 0, \quad \left(P^\alpha = \frac{M^{\alpha 4}}{u^4}\right), \tag{3·13a}$$

and then from (3·11)

$$\frac{dS^{\alpha\beta}}{ds} + u^\alpha P^\beta - u^\beta P^\alpha = 0. \tag{3·11a}$$

Both equations can, in this case, be integrated directly. The integral of (3·13a) expresses the conservation of energy and momentum,

$$P^\alpha = \text{const.},$$

while equation (3·11a) leads to the conservation of angular momentum,

$$S^{\alpha\beta} + X^\alpha P^\beta - X^\beta P^\alpha \equiv J^{\alpha\beta} = \text{const.}$$

Furthermore, one finds that only when $S^{\alpha\beta} = 0$ it will follow from (3·11a) and (3·13a) that $P^\alpha \propto u^\alpha$ and therefore $u^\alpha = \text{const.}$, i.e. X^α will be linear functions of t (or s). When $S^{\alpha\beta} \neq 0$ the co-ordinates X^α may be arbitrary functions of t. More characteristically, in the frame in which $P^i = 0$ the pole-dipole particle will in general have

$u^i \neq 0$. This fact can be expressed by saying that a pole-dipole particle can have an arbitrary 'internal motion'.

The simplest possible internal motion—uniform motion on a circle—has been discussed in detail. It turns out that this internal motion gives a most successful classical analogue of the microscopic motion, the 'Zitterbewegung', of the electron (Hönl & Papapetrou 1940). One might, therefore, expect that the internal motion will be of importance in the study of the elementary particles. On the contrary, for macroscopic particles the internal motion must be avoided: The motion must be entirely determined by the equations of motion. For this purpose one has to reduce the number of independent components $S^{\alpha\beta}$ to 3. The simplest possibility consists, for example, in assuming that in a certain system of co-ordinates it will be $S^{i4} = 0$ (equivalent to the assumption that the energy density $\mathfrak{T}^{44}$ is positive-definite).

4. Transformation properties

What remains to be shown is that the equations of motion (3·12), (3·13) can be written in a covariant form. For this purpose one has to prove first that the quantities $S^{\alpha\beta}$ are the components of a tensor, and secondly that the quantity

$$m = \frac{1}{u^4}(M^{\alpha 4} + \Gamma^{\alpha}_{\mu\nu} S^{\mu 4} u^{\nu})\, u_{\alpha} \tag{4·1}$$

is a scalar. These two theorems can be proved by making use of the transformation formulae of the quantities $M^{\lambda\mu\nu}$ and $M^{\mu\nu}$. The calculations leading to those formulae are quite simple but rather lengthy. For this reason only a description of the method which has been used will be given here, without the details of the calculations.

It is convenient to consider two types of transformations separately:

$$\text{(i)} \quad x^i = f^i(x'^{\mu}), \qquad x^4 = x'^4;$$
$$\text{(ii)} \quad x^i = x'^i, \qquad x^4 = f^4(x'^{\mu}).$$

An arbitrary transformation will be a superposition of one transformation (i) and one (ii).

The relation between the values $M^{\lambda\mu\nu}$ (in the system of co-ordinates x^{μ}) and $M'^{\lambda\mu\nu}$ (in the system x'^{μ}) can be found in the case (i) very easily, this being due to the fact that in both frames the integration extends over the same space. The hyperplane t = const. is identical with t' = const. The result is

$$M^{\lambda\alpha\beta} = \frac{\partial x^{\lambda}}{\partial x'^{\rho}} \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} M'^{\rho\mu\nu}. \tag{4·2}$$

In the case (ii) a more extensive calculation is needed, mainly because the hyperplanes t = const. and t' = const. are now different. The simplest transition from the first hyperplane to the second is by correlating those points of the two hyperplanes which have the same values of the co-ordinates x'^i and differ only in the values of x'^4. Furthermore, one must consider, in the first instance, infinitesimal transformations only. The result is

$$M^{\lambda\alpha\beta} = \frac{\partial x^{\alpha}}{\partial x'^{\mu}} \frac{\partial x^{\beta}}{\partial x'^{\nu}} \left(\frac{\partial x^{\lambda}}{\partial x'^{\rho}} - \frac{u^{\lambda}}{u^4} \frac{\partial x^4}{\partial x'^{\rho}}\right) M'^{\rho\mu\nu}. \tag{4·3}$$

One sees now that equation (4·2) is contained in (4·3). The additional terms in (4·3) vanish for a transformation of the type (i), since in this case $\partial x^4/\partial x'^i = 0$, while according to (3·5*a*) $M'^{4\mu\nu} = 0$. Furthermore, one can verify by a direct calculation that equation (4·3) has the group property: if it is valid for each of the transformations $x^\mu \to x'^\mu$ and $x'^\mu \to x''^\mu$, then it is valid also for the transformation $x^\mu \to x''^\mu$. One concludes that equation (4·3) is first valid for *any* infinitesimal transformation, and secondly, since a finite transformation can be decomposed into a series of infinitesimal ones, that equation (4·3) will be valid also for finite transformations. Thus equation (4·3) is the general transformation formula of the quantities $M^{\lambda\mu\nu}$.

In a similar way one can derive the transformation formula of the quantities $M^{\mu\nu}$. The final result, valid for infinitesimal as well as for finite transformations, is

$$M^{\alpha\beta} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} M'^{\mu\nu} - \left(\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\rho}\frac{\partial x^\beta}{\partial x'^\nu} + \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial^2 x^\beta}{\partial x'^\nu \partial x'^\rho}\right) M'^{\rho\mu\nu} + \frac{d}{ds}\left(\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}\frac{\partial x^4}{\partial x'^\rho}\frac{1}{u^4} M'^{\rho\mu\nu}\right). \tag{4·4}$$

Putting $\beta = 4$ in (4·3) one finds

$$M^{\lambda\alpha 4} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^4}{\partial x'^\nu}\left(\frac{\partial x^\lambda}{\partial x'^\rho} - \frac{u^\lambda}{u^4}\frac{\partial x^4}{\partial x'^\rho}\right) M'^{\rho\mu\nu}.$$

Substituting for $M'^{\rho\mu\nu}$ the value given by (3·8), a number of terms cancel and what remains is

$$M^{\lambda\alpha 4} = -\tfrac{1}{2}u^4 \frac{\partial x^\lambda}{\partial x'^\rho}\frac{\partial x^\alpha}{\partial x'^\mu} S'^{\rho\mu} + \frac{1}{2}\left(u^\lambda \frac{\partial x^\alpha}{\partial x'^\mu} + u^\alpha \frac{\partial x^\lambda}{\partial x'^\mu}\right)\frac{\partial x^4}{\partial x'^\rho} S'^{\rho\mu}.$$

On the right-hand side the first term is antisymmetric in λ, α, while the second is symmetric in the same indices. Hence

$$M^{\lambda\alpha 4} - M^{\alpha\lambda 4} = -u^4 \frac{\partial x^\lambda}{\partial x'^\rho}\frac{\partial x^\alpha}{\partial x'^\mu} S'^{\rho\mu}.$$

Or, because of (3·6*a*),

$$S^{\lambda\alpha} = \frac{\partial x^\lambda}{\partial x'^\rho}\frac{\partial x^\alpha}{\partial x'^\mu} S'^{\rho\mu}. \tag{4·5}$$

Thus it has been proved that $S^{\lambda\alpha}$ is a tensor.

Another conclusion drawn from (4·3) is worth mentioning. A particle cannot be a pole-dipole in one frame and a single-pole in another. If some of the quantities $M^{\lambda\alpha\beta}$ do not vanish in one frame, then the same will happen in any other frame. This result is a special case of the following general theorem. The order of the highest non-vanishing multipole of a particle is invariant against co-ordinate transformations.

Writing (4·4) with $\beta = 4$ one finds

$$M^{\alpha 4} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^4}{\partial x'^\nu} M'^{\mu\nu} - \left(\frac{\partial^2 x^\alpha}{\partial x'^\mu \partial x'^\rho}\frac{\partial x^4}{\partial x'^\nu} + \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial^2 x^4}{\partial x'^\nu \partial x'^\rho}\right) M'^{\rho\mu\nu} + \frac{d}{ds}\left(\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^4}{\partial x'^\nu}\frac{\partial x^4}{\partial x'^\rho}\frac{1}{u^4} M'^{\rho\mu\nu}\right). \tag{4·6}$$

In the right-hand side one can substitute $M'^{\mu\nu}$ according to (3·9). Then, considering (3·8) and the transformation formula for $\Gamma^\mu_{\rho\sigma}$,

$$\Gamma'^\mu_{\rho\sigma} = \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x^\epsilon}{\partial x'^\rho}\frac{\partial x^\zeta}{\partial x'^\sigma}\Gamma^\lambda_{\epsilon\zeta} + \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial^2 x^\lambda}{\partial x'^\rho \partial x'^\sigma},$$

one finds

$$-\frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^4}{\partial x'^\nu}\Gamma'^\mu_{\rho\sigma}M'^{\nu\sigma\rho} = \Gamma^\alpha_{\epsilon\zeta}u^\epsilon S^{4\zeta} + \frac{\partial^2 x^\alpha}{\partial x'^\rho\partial x'^\sigma}\frac{\partial x^4}{\partial x'^\nu}S'^{\nu\rho}u'^\sigma - \frac{\partial x^\alpha}{\partial x'^\mu}\Gamma'^\mu_{\rho\sigma}\frac{u^4}{u'^4}u'^\sigma S'^{4\rho}.$$

Similarly, because of (3·8),

$$\frac{\partial x^4}{\partial x'^\nu}\frac{\partial x^4}{\partial x'^\sigma}\frac{M'^{\sigma\mu\nu}}{u^4} = \frac{1}{2}\frac{\partial x^4}{\partial x'^\nu}\left(-S'^{\nu\mu} + \frac{u'^\nu}{u'^4}S'^{4\mu} + \frac{u'^\mu}{u'^4}S'^{4\nu}\right) = \frac{\partial x^4}{\partial x'^\nu}\frac{M'^{\nu\mu4}}{u'^4}.$$

Introducing these results in (4·6) one finally finds

$$\frac{1}{u^4}(M^{\alpha4} + \Gamma^\alpha_{\rho\sigma}u^\rho S^{\sigma4}) = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{1}{u'^4}(M'^{\mu4} + \Gamma'^\mu_{\rho\sigma}u'^\rho S'^{\sigma4}); \tag{4·7}$$

i.e. the quantity on the left-hand side is a vector.

From the last result it follows immediately that the quantity m defined by (4·1) is a scalar.

5. The covariant equations of motion

It is now possible to write the equations of motion in a covariant form. Introducing the quantities

$$\frac{DS^{\alpha\beta}}{Ds} = \frac{dS^{\alpha\beta}}{ds} + \Gamma^\alpha_{\mu\nu}S^{\mu\beta}u^\nu + \Gamma^\beta_{\mu\nu}S^{\alpha\mu}u^\nu, \tag{5·1}$$

which are the components of a tensor,* equation (3·12) is written in the form

$$\frac{DS^{\alpha\beta}}{Ds} + \frac{u^\alpha}{u^4}\frac{DS^{\beta4}}{Ds} - \frac{u^\beta}{u^4}\frac{DS^{\alpha4}}{Ds} = 0. \tag{5·2}$$

Multiplying (5·2) by u_β, then

$$\frac{1}{u^4}\frac{DS^{\alpha4}}{Ds} = u_\beta\frac{DS^{\alpha\beta}}{Ds} + \frac{u^\alpha}{u^4}u_\beta\frac{DS^{\beta4}}{Ds}.$$

Using this result one writes (5·2) in the final form

$$\frac{DS^{\alpha\beta}}{Ds} + u^\alpha u_\rho\frac{DS^{\beta\rho}}{Ds} - u^\beta u_\rho\frac{DS^{\alpha\rho}}{Ds} = 0. \tag{5·3}$$

This is the covariant formulation of the equation of motion of the spin.

To transform equation (3·13) one rewrites (3·9*a*) with the help of (3·6*b*) and (5·2):

$$M^{\alpha4} + \Gamma^\alpha_{\mu\nu}S^{\mu4}u^\nu = \frac{u^\alpha}{u^4}(M^{44} + \Gamma^4_{\mu\nu}S^{\mu4}u^\nu) + \frac{DS^{\alpha4}}{Ds}.$$

* Generally, when a tensor $A::$ is given on a line L, the quantity $\frac{DA::}{Ds}$ will be defined as follows. $A::$ is extended (arbitrarily) to a tensorfield in a small tube surrounding L. Then, denoting by $(A::)_{;\nu}$ the covariant derivative of this field, one writes

$$\frac{DA::}{Ds} = (A::)_{;\nu}u^\nu.$$

This formula shows directly that $\frac{DA::}{Ds}$ will be a tensor.

Multiplying by u_α/u^4 one gets, according to (4·1),

$$m = \frac{1}{(u^4)^2}(M^{44} + \Gamma^4_{\mu\nu} S^{\mu 4} u^\nu) + \frac{1}{u^4} u_\rho \frac{DS^{\rho 4}}{Ds}.$$

Hence
$$\frac{1}{u^4} M^{\alpha 4} = mu^\alpha - \Gamma^\alpha_{\mu\nu} S^{\mu 4} \frac{u^\nu}{u^4} + u_\beta \frac{DS^{\alpha\beta}}{Ds}.$$

Introducing this in (3·13), it is found, after some elementary calculations,

$$\frac{d}{ds}\left(mu^\alpha + u_\beta \frac{DS^{\alpha\beta}}{Ds}\right) + \Gamma^\alpha_{\mu\nu} u^\nu \left(mu^\mu + u_\beta \frac{DS^{\mu\beta}}{Ds}\right) + S^{\mu\nu} u^\sigma (\Gamma^\alpha_{\nu\sigma,\,\mu} + \Gamma^\alpha_{\mu\rho} \Gamma^\rho_{\nu\sigma}) = 0. \quad (5·4)$$

This relation is already covariant. To see it one has to consider first that if A^α is a vector, then

$$\frac{dA^\alpha}{ds} + \Gamma^\alpha_{\mu\nu} u^\nu A^\mu \equiv \frac{DA^\alpha}{Ds} \quad (5·5)$$

is also a vector; and secondly, that the curvature tensor of the basic field $g_{\mu\nu}$ is given by

$$R^\alpha_{\nu\sigma\mu} = \Gamma^\alpha_{\nu\sigma,\,\mu} - \Gamma^\alpha_{\mu\sigma,\,\nu} + \Gamma^\alpha_{\mu\rho} \Gamma^\rho_{\nu\sigma} - \Gamma^\alpha_{\nu\rho} \Gamma^\rho_{\mu\sigma}. \quad (5·6)$$

With (5·5) and (5·6) equation (5·4) takes the evidently covariant form

$$\frac{D}{Ds}\left(mu^\alpha + u_\beta \frac{DS^{\alpha\beta}}{Ds}\right) + \tfrac{1}{2} S^{\mu\nu} u^\sigma R^\alpha_{\nu\sigma\mu} = 0. \quad (5·7)$$

Equation (5·7) is a generalization of (2·10) in the sense that it reduces to (2·10) when $S^{\alpha\beta} = 0$. It is worth stressing that only in this simplest case the orbits will be geodesics of the field $g_{\mu\nu}$. When the particle has a non-vanishing spin *the orbits will differ from the geodesics*.

A detailed discussion of equations (3·12), (3·13) for a certain class of test particles moving in the Schwarzschild field will be published in a subsequent paper.

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