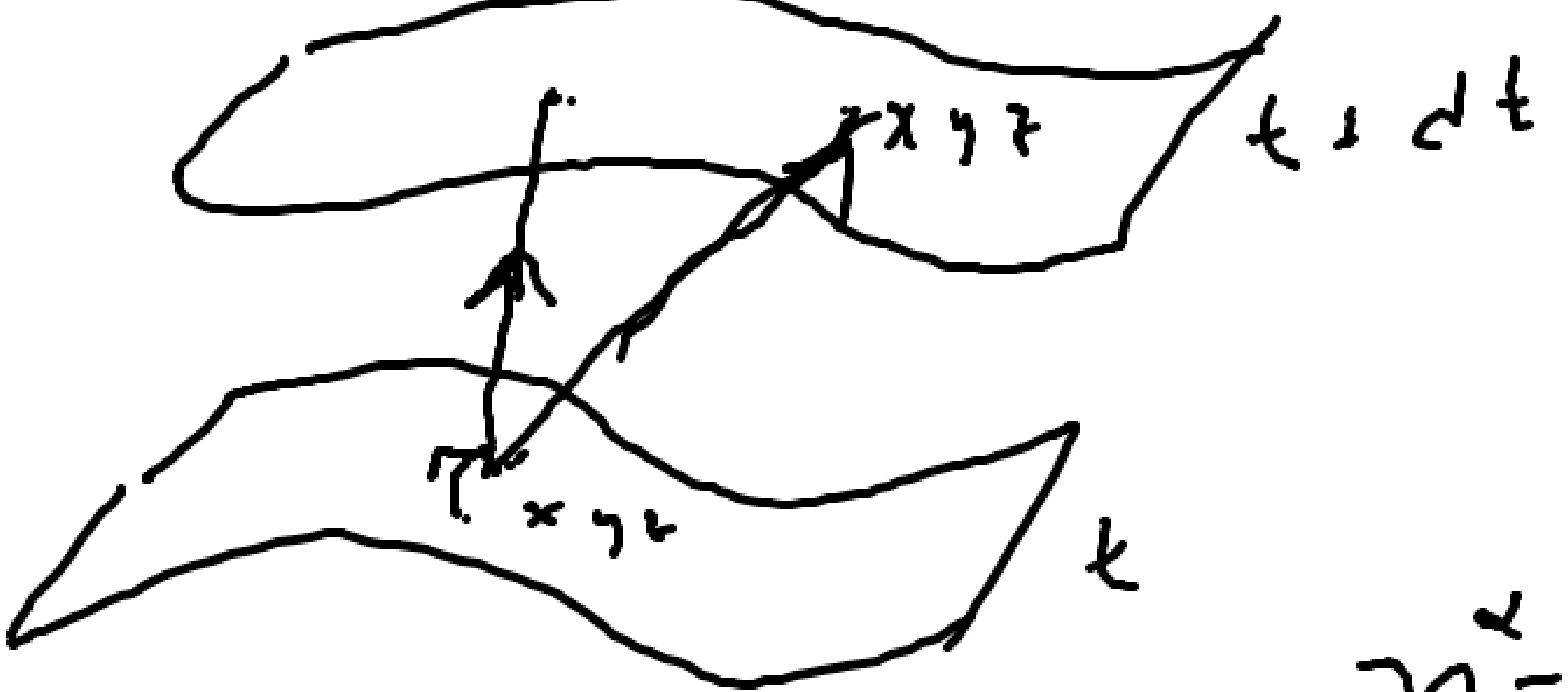
ADM Formalism



$\nabla_\alpha t$ ortho to hyper surface

$$n^\alpha = g^{\alpha\beta}\nabla_\beta t$$

$$n^\alpha n_\alpha = g^{\alpha\beta}\underbrace{\nabla_\alpha t}_{\delta_\alpha^t}\,\underbrace{\nabla_\beta t}_{\delta_\beta^t}$$

$$g^{tt} = -\frac{1}{\alpha^2}$$

$$n_\beta = \frac{1}{\alpha}\nabla_\beta t$$

$\underline{\gamma_{ij}} = g_{rj} \quad \beta_i = g_{ti}$

$$K_{ij} = (\nabla_\alpha \eta_\beta)\, \delta^\alpha_i\, \delta^\beta_j$$

$$\nabla_i \eta_j \qquad \eta_0 = \frac{1}{\alpha} \partial_0 t$$

$$= \frac{1}{\alpha}$$

$$= \Gamma^0_{ij}\, \eta_0$$

$$= \frac{1}{\alpha} \Gamma^0_{ij}$$

Rewrite Lag. $\mathcal{L} = \int \sqrt{-g}\, \underline{R}\, d^3x$

$$\sqrt{-g} = \alpha \sqrt{\gamma}$$

Problem

$$\sqrt{g}R = \int \sqrt{g}\, g\,\partial\partial g + \sqrt{g}\,\Gamma\Gamma \quad d^3x\,dt$$
$$\qquad\qquad\qquad\qquad \partial g\ \partial g$$

∂_0 Want 1st der in time.

$$\int g\,\partial_0^2 g \longrightarrow \int -\partial_0 g\,\partial_0 g$$

Rewrite R so only 1st deriv in time occur by doing int by parts in time.

Assume boundary integral is 0.

$$\pi^{ij} = \frac{\delta}{\delta \partial_t \gamma^{ij}} \int \underbrace{\sqrt{g}\, R}\, d^3x$$

only has $\partial_t \gamma_{ij}$

$$\pi^{ij} = \sqrt{\gamma} \left(\underbrace{\gamma_{kl} K^{kl}}_{K} \gamma_{ij} - K_{ij} \right)$$

Tensor density

$$K_{ij} \propto \partial_t \gamma_{ij}$$

Hamiltonian : $\int \left(\pi^{ij} \partial_t \gamma_{ij} - \sqrt{g}\, R \right) d^3x$

where I write $\partial_t \gamma_{ij}$ in terms of π^{ij}

$$\partial_t \gamma_{ij} = \frac{2\lambda}{\sqrt{\gamma}} \left(\pi_{ij} - \frac{1}{2} \pi \gamma_{ij} \right)$$

$$\pi = \gamma^{ij} \pi_{ij}$$

$$H = \int \alpha H^0 + \beta_i H^i \, d^3x$$

$$H^0 = 0 \qquad H^i = 0$$

$$H^0 = -\sqrt{\gamma}\left({}^{(3)}R + \frac{1}{\gamma}\left(\frac{\pi^2}{2} - \pi^{ij}\pi_{ij}\right)\right)$$

$$H^i = \left(2\left(\partial_j \pi^{ij} + \hat{\Gamma}^i_{jk}\,\pi^{jk}\right)\right)$$

π^{ij} is not a tensor

tensor density

$$H^i = 2\,\hat{\nabla}_j \pi^{ij}$$

$$\mathcal{H} = \int (\alpha H^0 + \beta_i H^i)\, d^3x$$

$$H^0 = 0 \,,\quad H^i = 0 \implies \mathcal{H} = 0$$

For solutions, the Energy is 0.

H^0, H^i have no time deriv.

So they are not Evolution eqns

They are constraints on initial data.

$$\{A, B\} \qquad q_i,\ p_i$$

$$\hookrightarrow \sum_i \left(\frac{\partial A}{\partial q_i} \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \frac{\partial B}{\partial q_i} \right)$$

$$A = q_1 \qquad B = H$$

$$\{q_1, H\} = \frac{\partial H}{\partial p_1} = \partial_t q_1$$

$$\text{If } \gamma_{ij},\ (\cancel{\xi_0 H^0} + \xi_i H^i) = \underbrace{\mathcal{L}_{\xi^i} \gamma_{ij}}_{\text{inf coord transf.}}$$

$$\{\gamma_{ij},\ \xi_0 H^0 + \xi_i H^i\} = \text{inf spacetime coord trans}$$

on solutions.

γ_{ij} obeys field eqns.

$$\left\{\gamma_{ij}, \int \xi_0 H' + \xi_i H_i\right\} = \text{coord trans. of } \gamma_{ij}$$

But Ham. eqns. are then coord transformations.

??

In Q.Th.

H^0 an H^i functions of operators γ_{ij}, π^{ij}

cannot make H^0 as functn of operator $= 0$.

γ_{ij}, π^{ij} conjugate op on some big Hilbert space +

Physical Hilbert space

$$H^0 |\psi_p\rangle = H^i |\psi\rangle_{phys} = 0.$$

Problem:

Physical operators A_{ph}.

$$A|\psi_p\rangle \rightarrow |\phi_p\rangle$$

$$[A, H_0] = [A, H_i] = 0$$

$$f(H^0, H^i)\ ?$$

Eqns. motion

$$\frac{\partial A}{\partial t} = [A, \mathcal{H}] = [A, \int \alpha H^0 + \beta_i H^i] = 0$$

All physical operators must be

time indep. (Constant of motion)

Problem of Time

Time is relational(?)

$$\underbrace{X = X_0 + PT}_{\text{Current Problem.}}\ !\ ??$$

$H^0(x) = H^i(x) = 0$

If one represent $\pi^{ij}(x) = i \dfrac{\partial}{\partial \gamma_{ij}(x)}$ into H^0, H^i

$$H\left(\frac{\partial}{\partial \gamma_{ij}}, \gamma_{ij}\right) \Psi(\gamma_{ij}) = 0$$

Wheeler-DeWitt equation.

$$H^0 |\Psi_p\rangle = H^i |\Psi_p\rangle = 0.$$

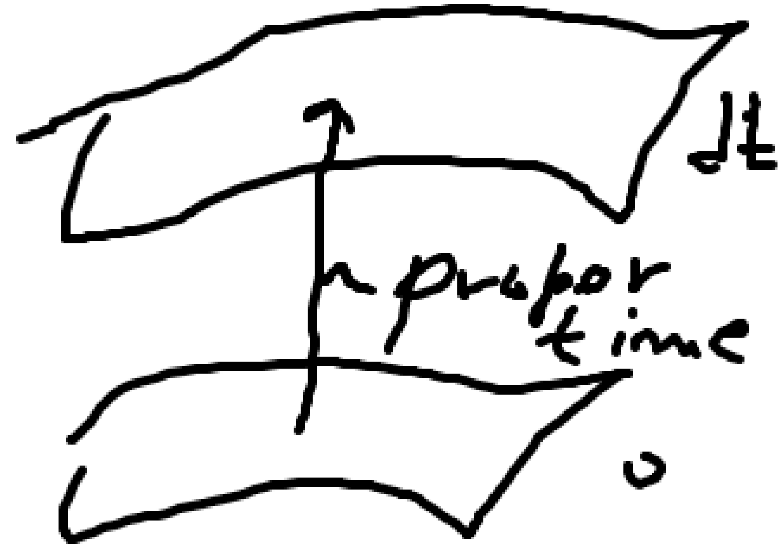
$$H^0 = H^i = 0$$

Coord freedom.

Coord Conditions. ?

$$\alpha = 1 \,, \quad \beta_i = 0$$

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & \gamma_{ij} & \\ 0 & & & \end{pmatrix}$$



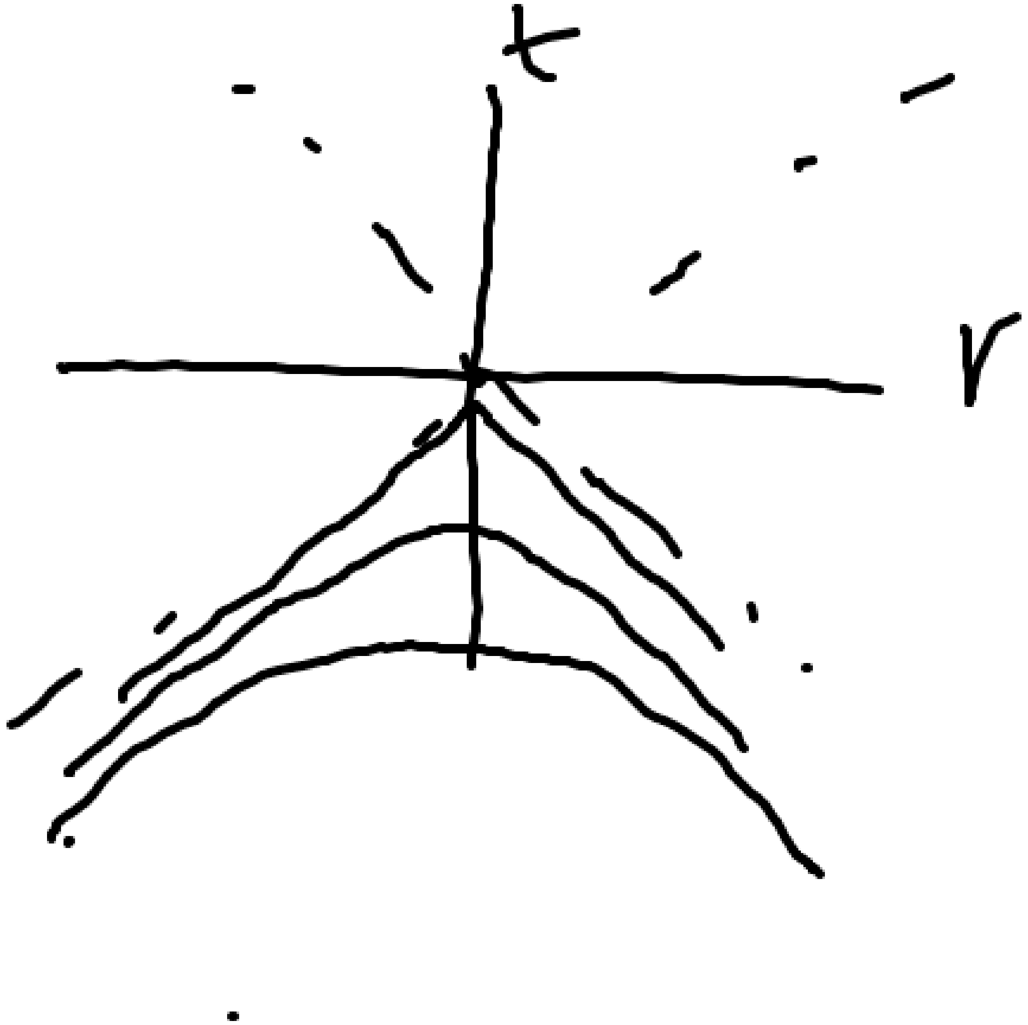
t is proper time.

$$ds^2 = -dt^2 + \gamma_{ij}\, dx^i dx^j$$

Gaussian Normal coord.

$$d\tau^2 = -dt^2 + dr^2 + r^2 d\Omega^2$$

$$t = \tau \cosh\rho$$
$$r = \tau \sinh\rho$$
$$\Rightarrow \quad ds^2 = -d\tau^2 + \tau^2 d\rho^2 + \tau^2 \sinh^2(\rho)\, d\Omega^2$$



$\tau = 0$ is a problem.

At $\tau = 0$ both intrinsic & Ext curvature blow up.

Gaussian Normal coord are (almost) always pathological (coord singularities)

$$H^0 = H^i = 0$$

$$\partial_t H^0 = \partial_t H^i = 0$$

$$\partial_t H^0 = a H^0$$

If $H^0 = \epsilon$

$$H^0 = \epsilon e^{at}$$

with "wrong" curd choice
Eqns are unstable.

Unsdable

$$H^0 = \frac{1}{\sqrt{\gamma}}\left(\frac{\pi^2}{2} - \pi_{ij}\pi^{ij}\right) + \sqrt{\gamma}\,{}^3R$$

$$\tilde{\pi}_{ij} = \left(\pi_{ij} - \frac{1}{3}\pi\gamma_{ij}\right) \qquad \gamma_{ij}\gamma^{ij} = 3$$

$$H^0 = \frac{1}{\sqrt{\gamma}}\left(\underbrace{\tilde{\pi}_{ij}\tilde{\pi}^{ij}}_{\text{positive}} - \underbrace{\frac{1}{6}\pi^2}_{\text{negative K.E.}}\right)$$

$$-p^2 + q^2$$

$\pi = 0 \qquad K = 0 \Rightarrow$ coord system chosen so coord volumes do not change physicall

Is there acoord system
which is stable?

Yes. Frans Pretorius.

Linear Theory.

$$\partial_i \bar{h}^{ij} = 0$$

Harmonic Coord.