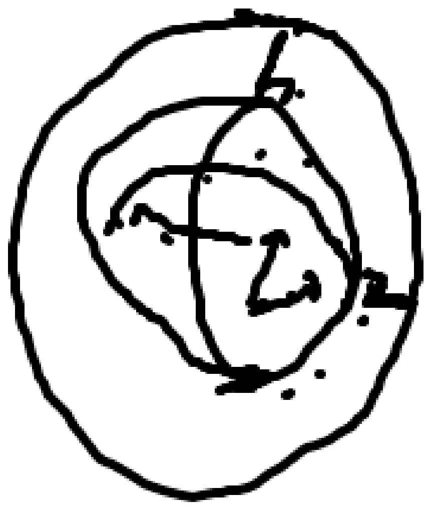
Spher. Symmetry.



spher coord.

$r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$

$2\pi r =$ Circ.

$4\pi r^2 =$ Area.

r, θ, ϕ

u = "time"

u: null coord.

$ds^2 = f(r)$

$ds^2 = \ldots 0 \, dr^2 - \ldots$

$dr \, d\theta, \ dr \, d\phi, \ du \, d\theta \ \ du \, d\phi \Rightarrow$ all 0

$ds^2 = f(r) du^2 + 2g(r) \, du \, dr - r^2 \underbrace{(d\theta^2 + \sin^2\theta \, d\phi^2)}_{d\Omega^2} \Big\} \, d\Omega^2$

g^{ij} $\quad \partial_k g_{ik}$

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl}\left(\partial_i g_{lj} + \partial_j g_{li} - \partial_l g_{ij}\right)$$

$$S \cancel{ds} = \int g_{ij} \frac{dx^i}{ds}\frac{dx^j}{ds}\, ds \qquad \delta x^k$$

$$\frac{d^2}{ds^2}x^i + \Gamma^i_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds}$$

$$f\left(\frac{du}{ds}\right)^2 + 2g(r)\frac{dr}{ds}\frac{du}{ds} - r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$$

$$-\frac{d}{ds}\left(2f\frac{du}{ds}\right) - 2\frac{d}{ds}\left(g\frac{dr}{ds}\right) = 0$$

$$-2\frac{d}{ds}\left(g\frac{du}{ds}\right) + f'\left(\frac{du}{ds}\right)^2 - 2g'\frac{du}{ds}\frac{dr}{ds}$$

$$-2r\left(\frac{d\theta}{ds}\right)^2 - 2r\sin^2\theta\left(\frac{d\phi}{ds}\right)^2$$

$$-2g\frac{d^2u}{ds^2} - 2g'\cancel{\frac{du}{ds}\frac{dr}{ds}} - 2g'\cancel{\frac{du}{ds}\frac{dr}{ds}}$$

$$+ f'\left(\frac{du}{ds}\right)^2 - 2r\left(\frac{d\theta}{ds}\right)^2 - 2r\sin^2\theta\left(\frac{d\phi}{ds}\right)^2$$

$$\Gamma^u_{uu} = -\frac{f'}{2g} \qquad \Gamma^u_{\theta\theta} = \frac{r}{g} \qquad \Gamma^u_{\phi\phi} = \frac{r\sin^2\theta}{g}$$

$$R_{ij}{}^{k}{}_{l} = \partial_i \Gamma_{jl}^{k} - \partial_j \Gamma_{il}^{k} + \Gamma\Gamma - \Gamma\Gamma$$

If $G_{ij} = 0$ Then $R_{ij} = 0$

$$R_{ij} = G_{ij} - \frac{1}{2} g^{kl} G_{kl}\, g_{ij}$$

$$R_{rr} \propto \frac{g'}{g} = 0 \qquad g = \text{const.}$$

$$d\bar{u} = |g|\, du$$

$$f\, du^2 + 2g\, du\, dr$$

$$= \frac{f}{g^2}\, d\bar{u}^2 + 2(\pm)\, d\bar{u}\, dr$$

$R_{\theta\theta} \propto \underline{r f' + f - g^2}$ $\quad (r.f')' = g$

$R_{\theta\theta} = 0 \quad f' = g^2 + \frac{2M}{r} \quad f' = \frac{gr + 2m}{r}$

$f = 1 + \frac{2m}{r}$

$f = 1 - \frac{2m}{r}$

far from $r = 0$

$1 - \frac{2Gm}{c^2 r}$

$$ds^2 = \left(1 - \frac{2m}{r}\right) du^2 \pm 2\,du\,dr \boxed{- r^2 d\Omega^2}$$

$$\left(1 - \frac{2M}{r}\right)\left(du\left(du + 2\frac{dr}{1 - 2\frac{M}{r}}\right)\right) = \left(1 - \frac{2m}{r}\right)\left(\left(du + \frac{dr}{\left(1 - \frac{2m}{r}\right)}\right)^2 - \frac{dr^2}{\left(1 - \frac{2m}{r}\right)^2}\right)$$

$$dt = du + \frac{dr}{1 - 2\frac{m}{r}} \quad t = u + r^*$$

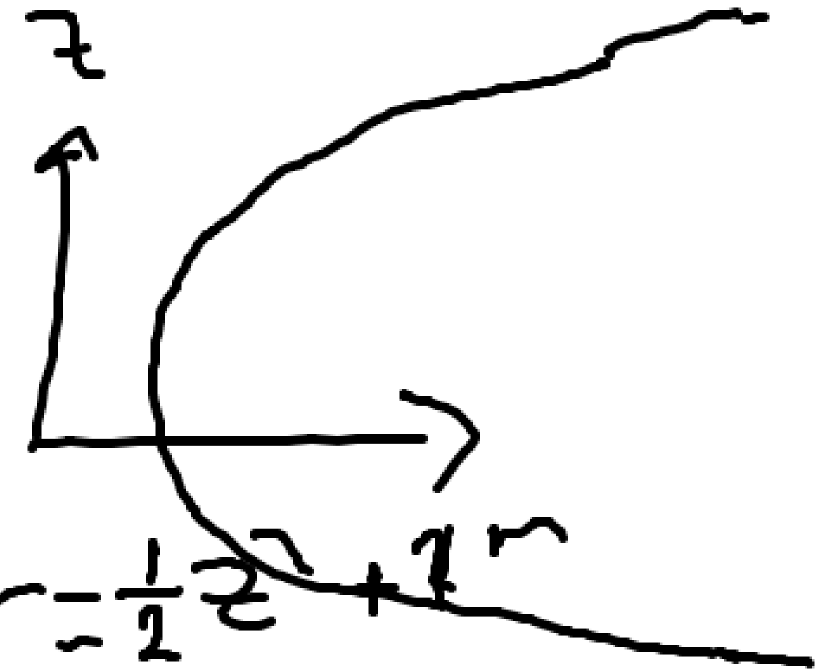
$$r^* = \int \frac{dr}{1 - 2m/r}$$

$$ds^2 = \underbrace{\left(1-\frac{2m}{r}\right)}_{\to 0} dt^2 - \underbrace{\frac{dr^2}{1-\frac{2m}{r}}}_{\text{diverges}} - r^2 d\Omega^2$$

What is $r=2m$?

1911 Flamm.

$$\frac{dr}{1-\frac{2m}{r}} + r^2 d\Omega^2$$



$$\underline{r = \tfrac{1}{2}z^2 + 2m}$$

$$dz^2 + dr^2 + r^2 d\Omega^2$$

$$dr = 2z\,dz \qquad dz = \frac{dr}{2\sqrt{r-1}}$$

$$\frac{dr^2}{4(r-1)} + dr^2 = dr^2\left(\frac{r}{r-1}\right) = \frac{dr^2}{1-\frac{1}{r}}$$

1921, 22 Painlevé, Gullstrand

$$\left(1-\frac{2M}{r}\right)dt^2 \pm 2\sqrt{\frac{2M}{r}}\,dt\,dr - dr^2 - r^2 d\Omega^2$$

$$\left(1-\frac{2m}{r}\right)\left(d\tau + \frac{f\,dr}{1-2M/r}\right)^2 - \frac{dr^2}{1-\frac{2m}{r}} - r^2 d\Omega^2$$

$$= \left(1-\frac{2m}{r}\right)d\tau^2 + \frac{2f}{r} + \underbrace{\left(\left(1-\frac{2m}{r}\right)f^2 - \frac{1}{1-\frac{2m}{r}}\right)}_{=1} dr^2$$

choose f,

$$f = \sqrt{\frac{2M}{r}}$$

Lanzos $\rightarrow$ coord singularities could make metric singular.

$$\begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & r^2 & 0 \\ & & 0 & r^2 \end{pmatrix}$$

$y, x \quad y = \ln x$

$$dy^2 = \left(\frac{dx}{x}\right)^2$$

$$\left(1-\frac{2m}{r}\right)(du^2) + 2\,du\,dr - r^2 d\Omega^2$$

$u = t - r^* \qquad v = t + r^*$

$$\left(1-\frac{2m}{r}\right)dv^2 - 2\,dv\,dr - r^2 d\Omega^2$$

Kruscal coords. (1960)