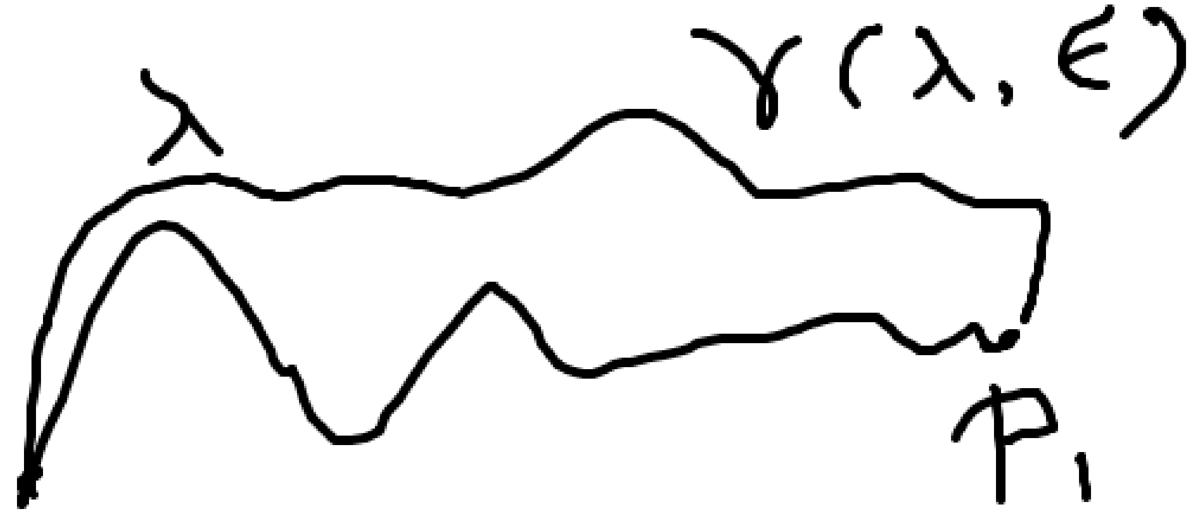


$\gamma(\lambda,0) = \gamma(\lambda)$

P_0

$\gamma(\lambda_0) = P_0 \quad \gamma(\lambda_1) = P_1$

λ_0, λ_1 are fixed.

$ds^2 = dt^2 - dx^2 - \cdots$

st line → Extremum

$$S = \sqrt{\sum_{ij} g_{ij}(\gamma(\lambda,\epsilon)) \frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}} \qquad g_{ij} = g_{AB}\,\partial_i^A \partial_j^B$$

$$\partial_\gamma^A = \sum_i \frac{dx^i}{d\lambda}\partial_i^A$$

$$\frac{d}{d\epsilon}\int_{\lambda_1}^{\lambda_2} S\, d\lambda = 0$$

$$g(\gamma(\lambda,\epsilon)) = g(x^\alpha(\lambda,\epsilon))$$

$$\sum_j \int \frac{1}{2}\left(\frac{d}{d\lambda}\frac{\partial g_{ij}}{\partial s}\frac{dx^j}{d\lambda} - \sum_k \frac{\partial_i g_{jk}}{\partial s}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}\right)\frac{dx^i}{d\epsilon}\, d\lambda$$

$$+ \cancel{g_{ij}\frac{dx^j}{d\lambda}\frac{dx^i}{d\epsilon}}\Bigg|_{\lambda=\lambda_0}^{\lambda_1}$$

$$0$$

$$0 = \sum_j \frac{d}{d\lambda}\left(\frac{g_{ij}}{s}\frac{dx^j}{d\lambda}\right) - \frac{1}{2}\sum_{jk}\left(\frac{\partial_i g_{jk}}{s}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}\right)$$

Geodesic Eqn.

$$\frac{g_{ij}}{s}\frac{d^2x^j}{d\lambda^2} + \frac{\partial_k g_{ij}}{s}\frac{dx^k}{d\lambda}\frac{dx^j}{d\lambda} - \frac{1}{2}\frac{\partial_i g_{jk}}{s}\frac{dx^j}{d\lambda}\frac{dx^k}{d\lambda}$$

$$+ \cancel{(\ )\frac{ds}{d\lambda}} \to 0$$

Choose λ so that $S = 1$

$$S = \sqrt{g_{ij}\frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}}$$

$s(\lambda)$

$$= \sqrt{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\left(\frac{ds}{d\lambda}\right)^2}$$

$$= \sqrt{\phantom{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}}}\;\frac{ds}{d\lambda}$$

$S = 1$ along $\gamma(\lambda, 0)$ only.

Def special λ by s

$$0 = \sum_j g_{ij} \frac{d^2 x^j}{ds^2} + \frac{\sum_{jk}}{2} \underbrace{\left(\partial_j g_{ik} + \partial_k g_{ij} - \partial_i (g_{jk}) \right)}_{\substack{\text{Christofel symbol} \\ \text{of 1st kind.} \\ \text{coord dep. set of} \\ \text{numbers.}}} \frac{dx^j}{ds} \frac{dx^k}{ds}$$

St lines $\Longrightarrow$ coord system
solve eqns.

$$\frac{d}{d\epsilon}\int \sqrt{\underbrace{g_{AB}\,\dot{\partial}_\gamma^{A}\,\dot{\partial}_\gamma^{B}}}\,d\lambda = 0$$

Example : ρ, θ

$$g_{\rho\rho} = 1, \quad g_{\theta\theta} = \rho^2 \quad g_{\rho\theta} = g_{\theta\rho} = 0$$

$$ds^2 = d\rho^2 + \rho^2 d\theta^2 \qquad r(\lambda,\epsilon),\ \theta(\epsilon)$$

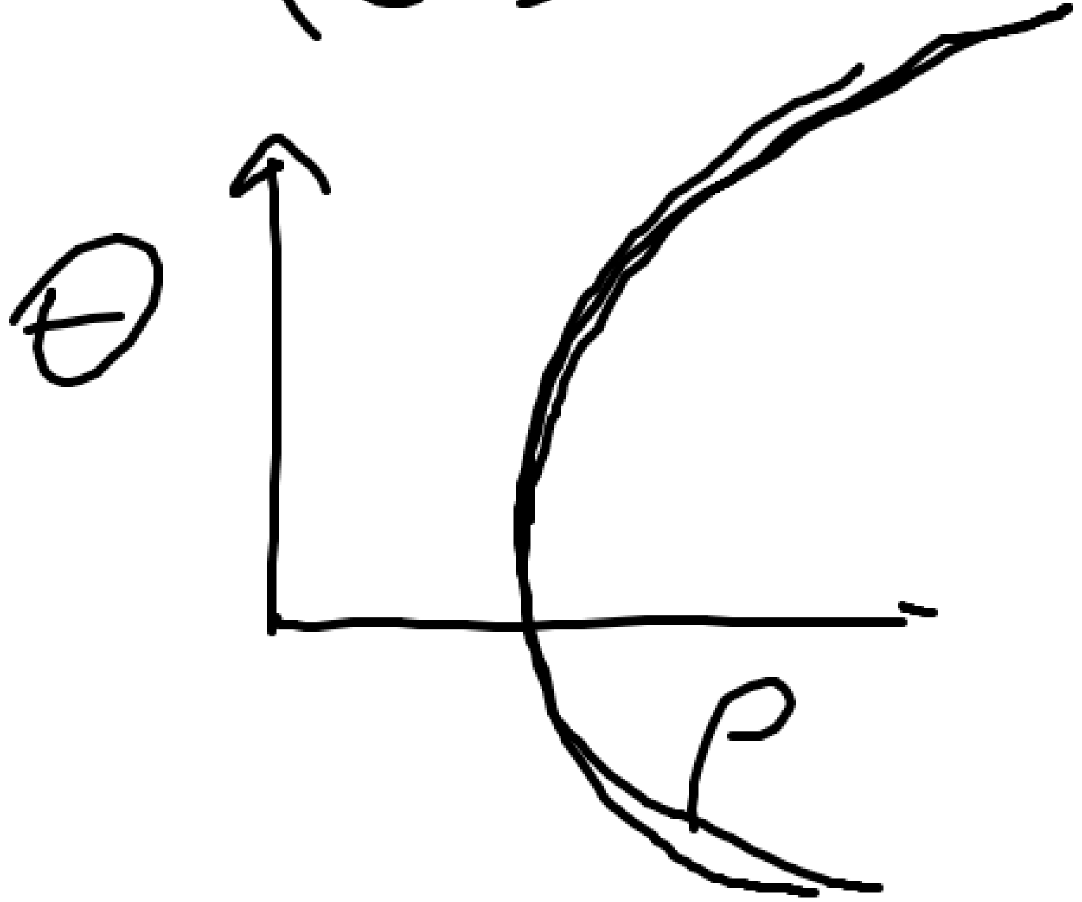
$$\frac{d}{d\epsilon}\int S\,d\lambda = 0 \qquad \lambda = s \quad : \quad \frac{d}{ds}\left(g_{\rho\rho}\frac{dr}{ds}\right) - 2r\left(\frac{d\theta}{ds}\right)^2 = 0$$

$$\frac{d}{ds}\left(r^2\frac{d\theta}{ds}\right) = 0$$

$$S = \left(\frac{d\rho}{ds}\right)^2 + \rho^2\left(\frac{d\theta}{ds}\right)^2 = 1$$

$$\rho^2 \frac{d\theta}{ds} = L \qquad \frac{d\theta}{ds} = \frac{L}{\rho^2}$$

$$\left(\frac{d\rho}{ds}\right)^2 = \left(1 - \frac{L^2}{\rho^2}\right) \qquad ds = \frac{\rho\, d\rho}{\sqrt{(\rho^2 - L^2)}}$$



$$s = \sqrt{\rho^2 - L^2}$$

$$\rho^2 = L^2 + s^2$$

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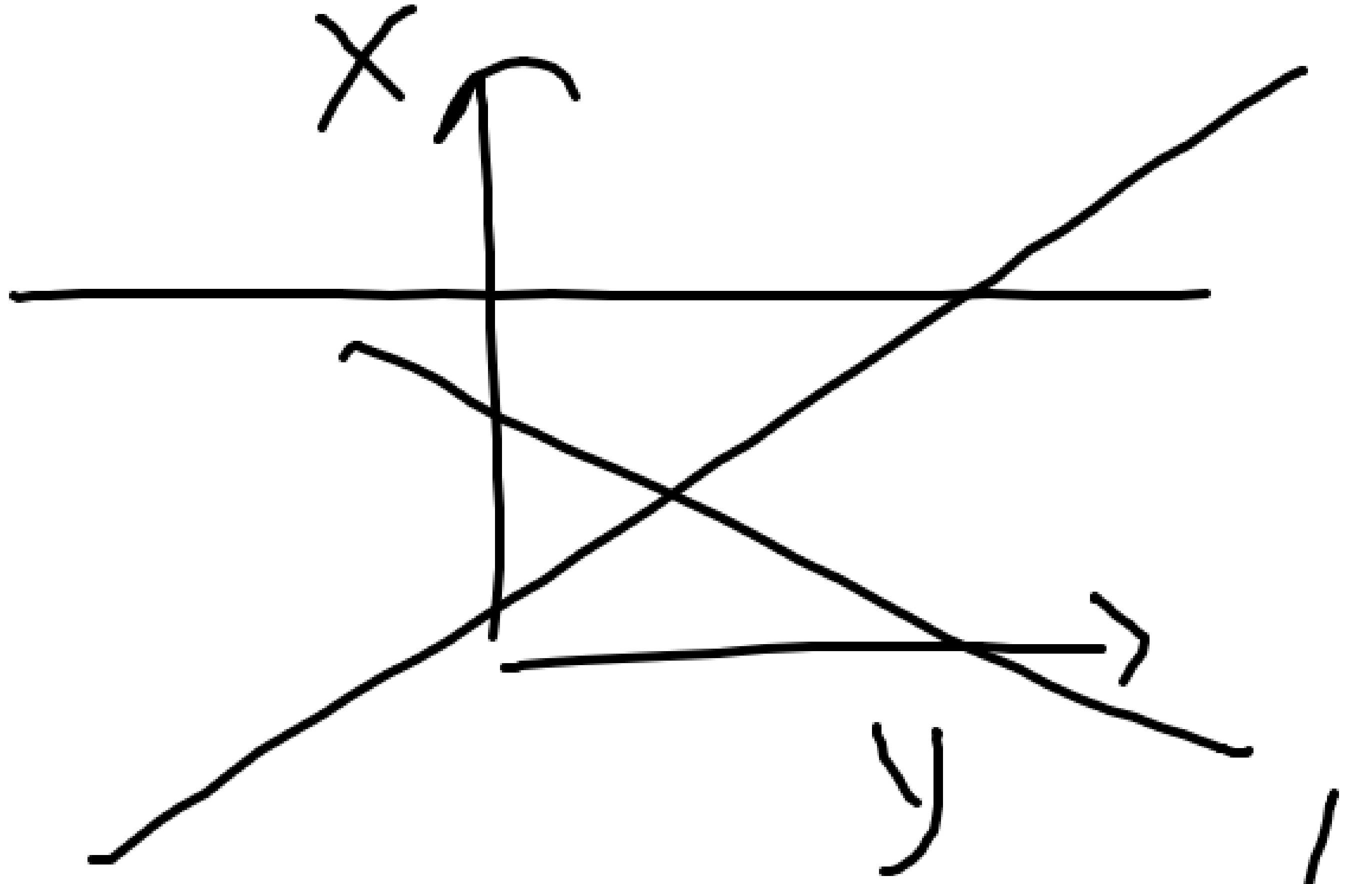
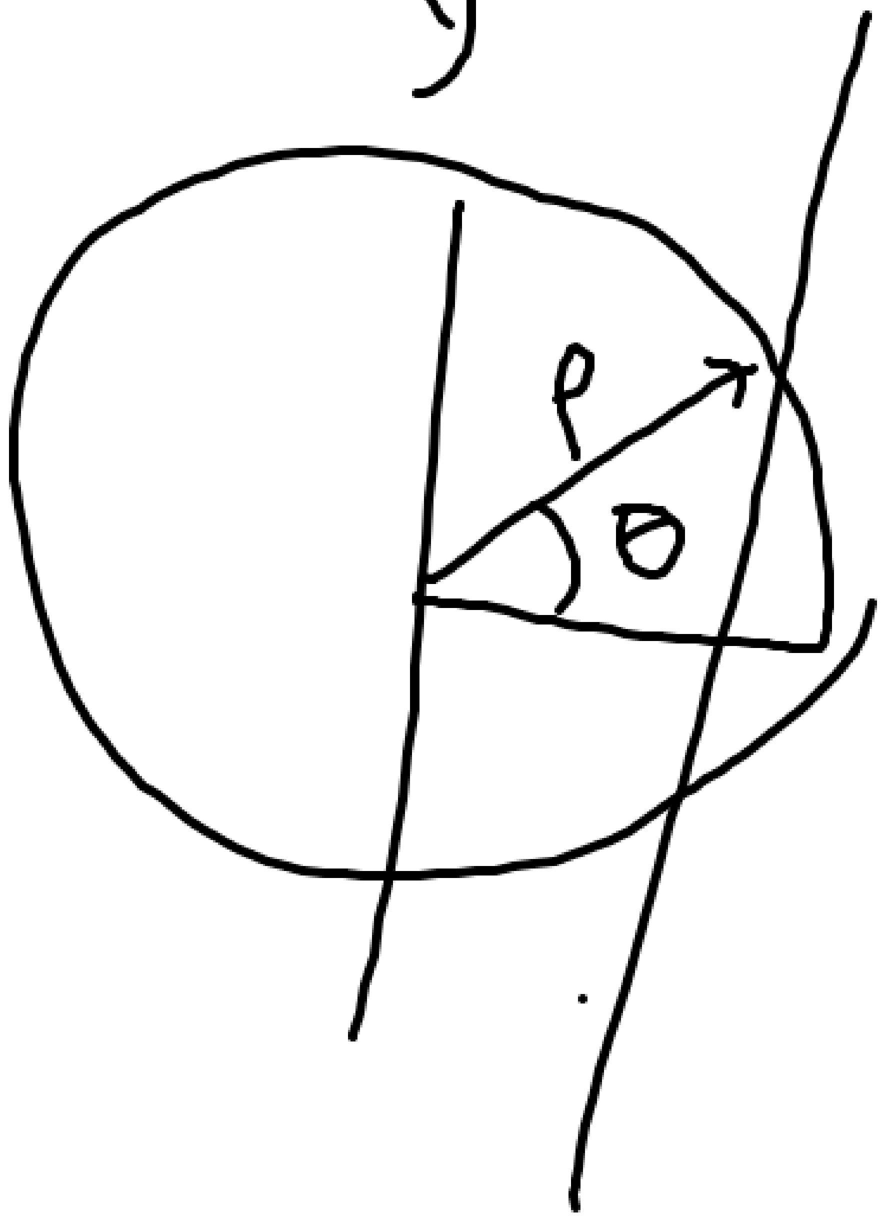
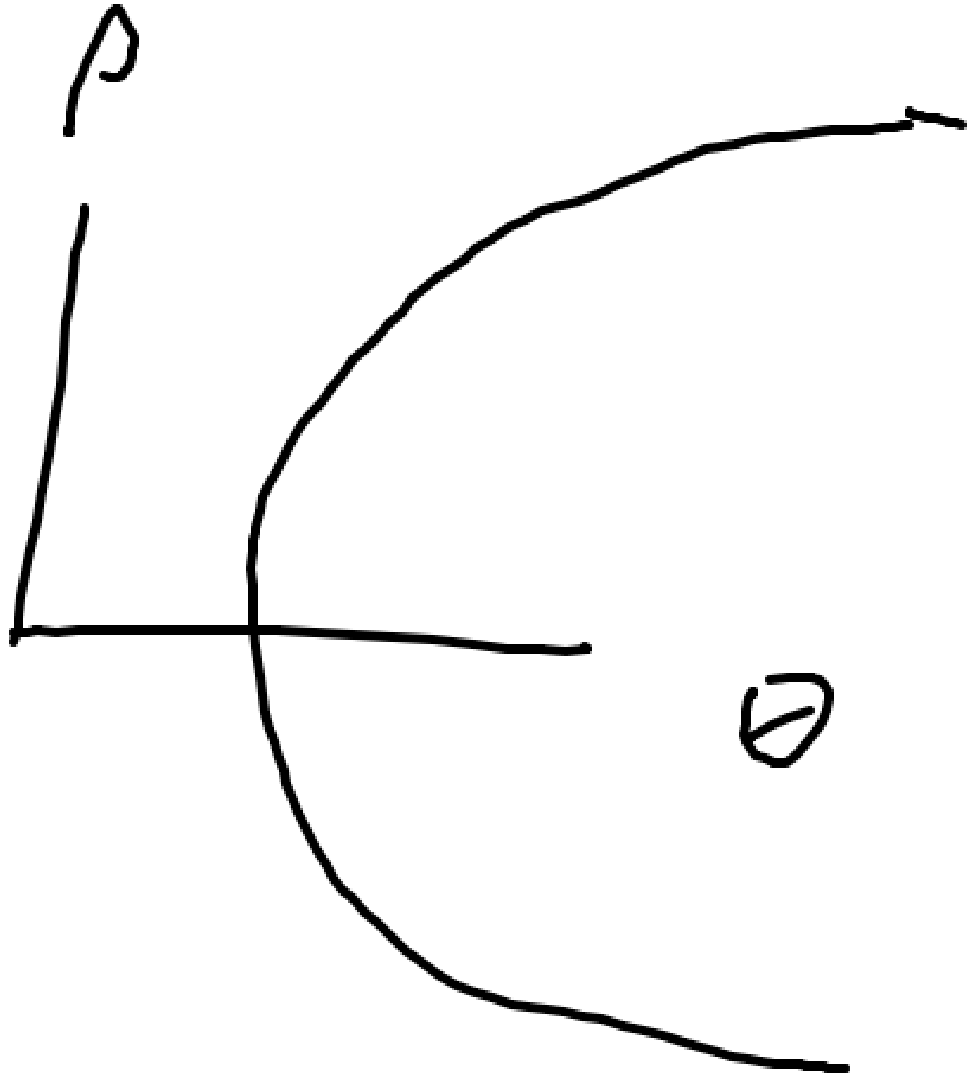


$$x = \rho \cos\theta$$
$$y = \rho \sin\theta$$





Tangent vectors. – lengths.

cotang vectors. ; $\ell^2(W_A, U_B)$

$$g_{AB} V^A W^B \qquad \text{Fix } V^A$$

$$(g_{AB} V^A)(W^B) = \#$$

$$U_B W^B = \# \qquad \text{If for all } W^B$$

$$V^A \to U_B$$

$$\underline{U_B W^B} = \underline{g_{AB} V^A W^B}$$

$$U_B = g_{AB} V^A \qquad \text{Map } V^A \to U_B.$$

$$\hat{g}^{AB} U_A R_B = g_{AB} V^A W^B$$

where $U_B = g_{AB} V^A$

$R_B = g_{AB} W^A$

$\hat{g}^{AB} \underline{U_A}$ is a tang vector components.

$$\{\hat{g}^{ij}\} = \{g_{ij}\}^{-1}$$

matrix

$$\hat{g}^{ij} g_{kj} = \hat{g}^{ij} g_{jk} = \delta^i{}_k$$

$\hat{g}^{AB}$ → metric on cotang vector assoc with tang vectors.

g_{AB} on tang vectors.

$$g_{AB} V^A V^B = 0$$

$$U_B = g_{AB} V^A = df_A$$

$$df_A V^A = 0$$

Tang V^A lies inside surface of f.

Inverse metric

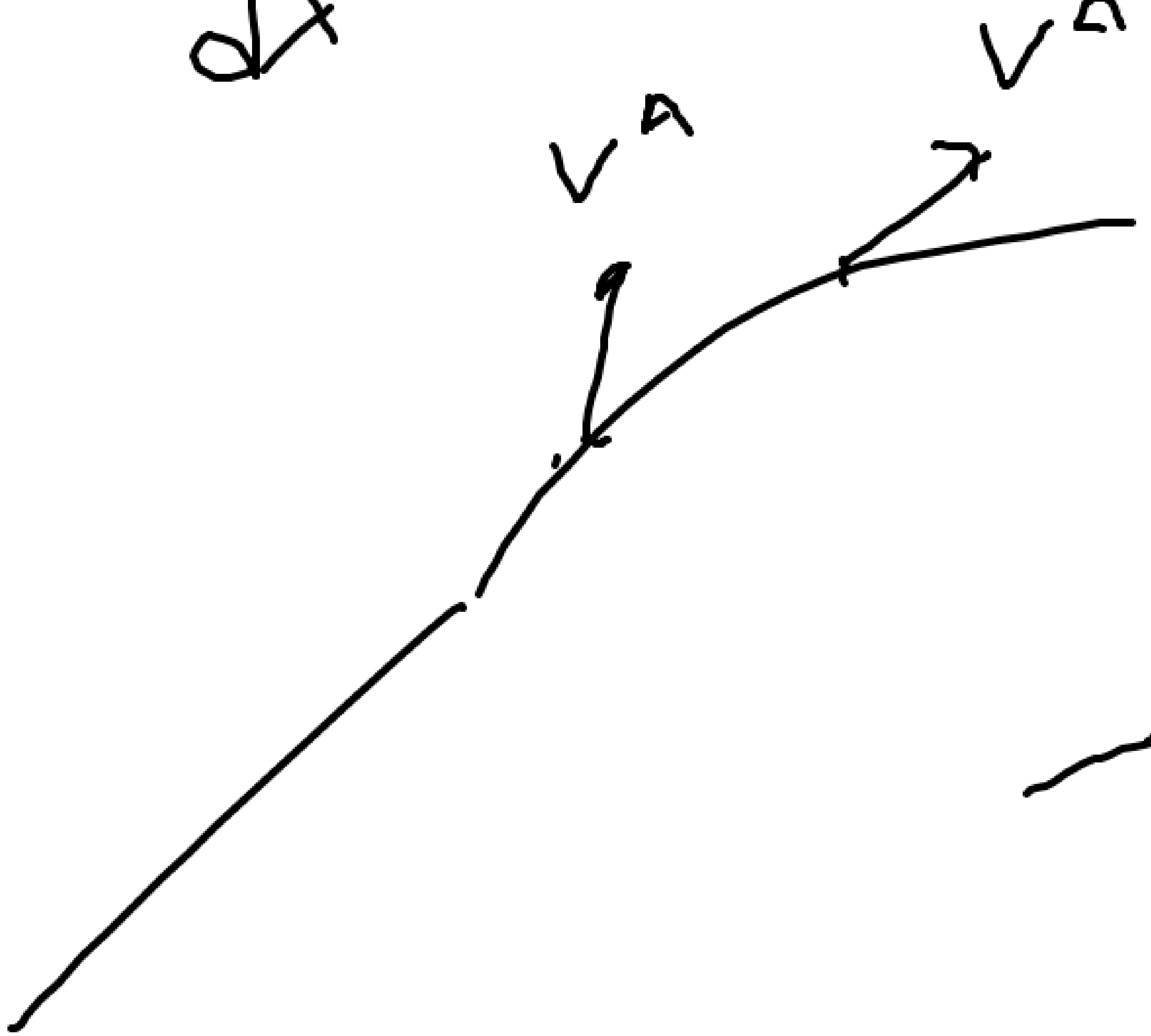
$$\Gamma_{kij} = \frac{1}{2}\left(\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ij} \right)$$

$$\Rightarrow \Gamma^{\ell}{}_{ij} = \sum_k g^{\ell k}\left(\frac{1}{2}\partial_i g_{jk} + \frac{1}{2}\partial_j g_{ik} - \partial_k g_{ij} \right)$$

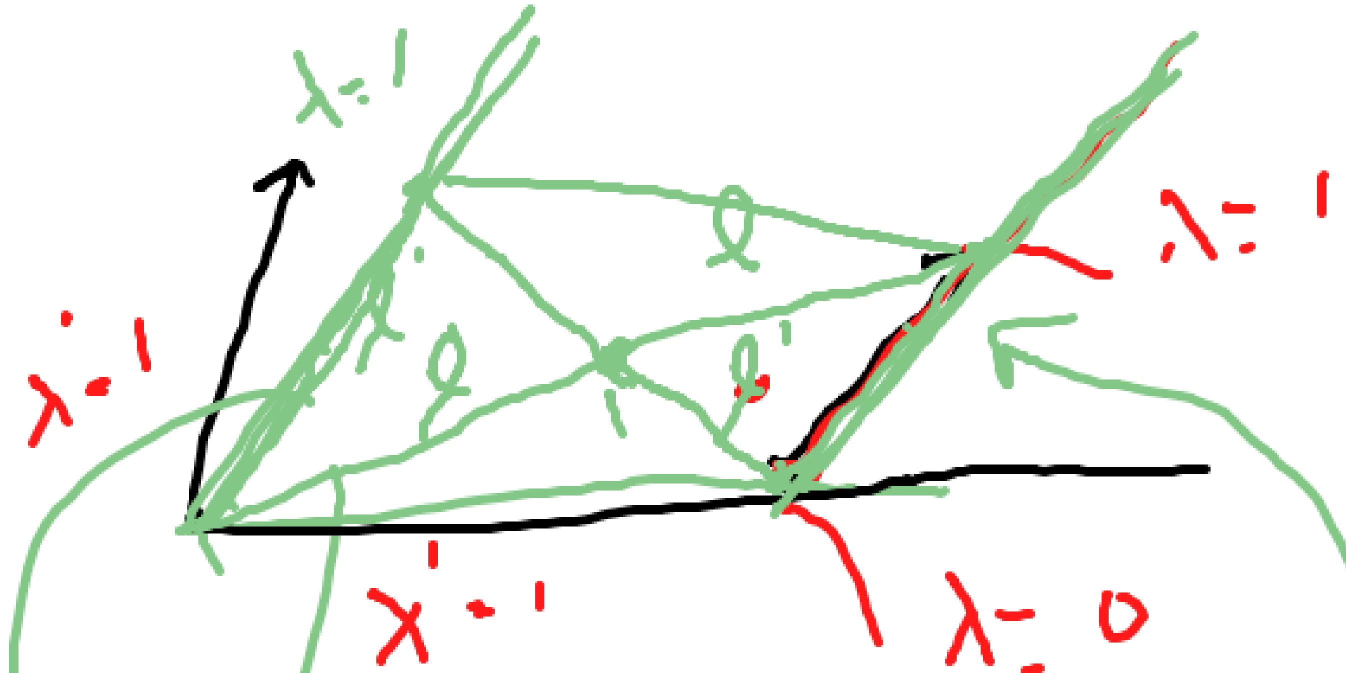
Christ. symbol of 2nd kind

Derivatives.

$$\frac{d}{d\lambda} V^A(\gamma(\lambda)) = \lim_{\epsilon \to 0} \frac{V^A(\gamma(\lambda+\epsilon)) - V^A(\gamma(\lambda))}{\epsilon}$$

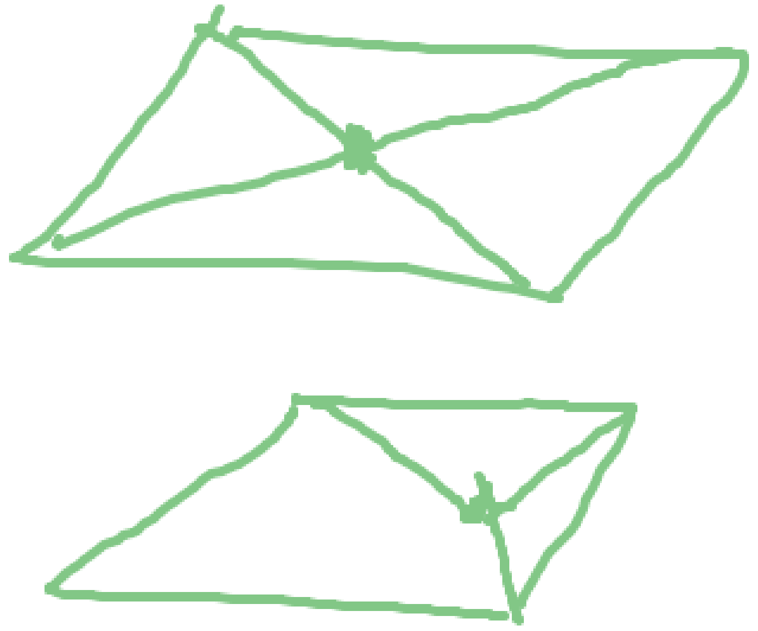


Parallel transport



$$\lim_{\epsilon \to 0} \frac{P(V^A(\lambda+\epsilon)) - V^A(\lambda)}{\epsilon}$$

$\frac{1}{2}\ell$.

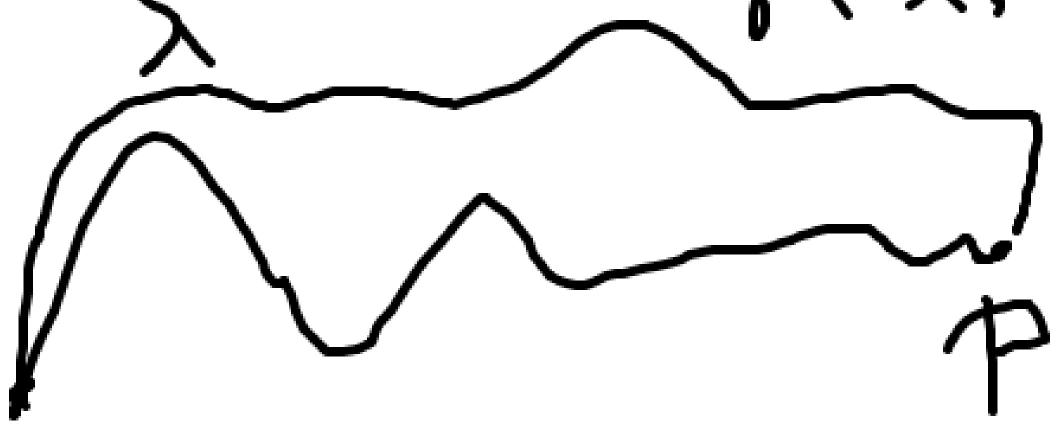


tang vect is // to

Indep of coord.

Physical

$\longrightarrow$ $\dfrac{DV^A}{D\lambda}$ = Parallel derivative



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$\Downarrow$

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$$\frac{d}{d\epsilon} \int_{\lambda_1}^{\lambda_2} S \, d\lambda = 0 \qquad \partial_\gamma^A = \sum_i \frac{dx^i}{d\lambda} \partial_i^A$$

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$\cancel{S}$ $s(\lambda)$

$$= \sqrt{g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{ds}{d\lambda}^2}$$

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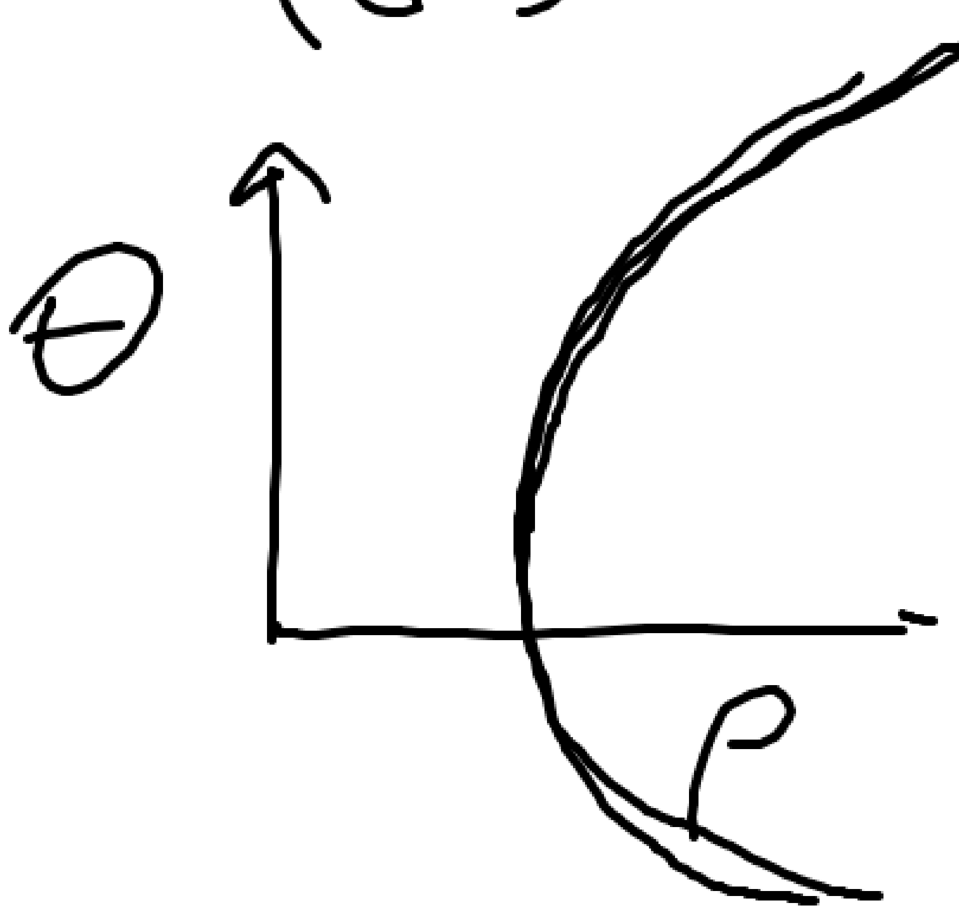

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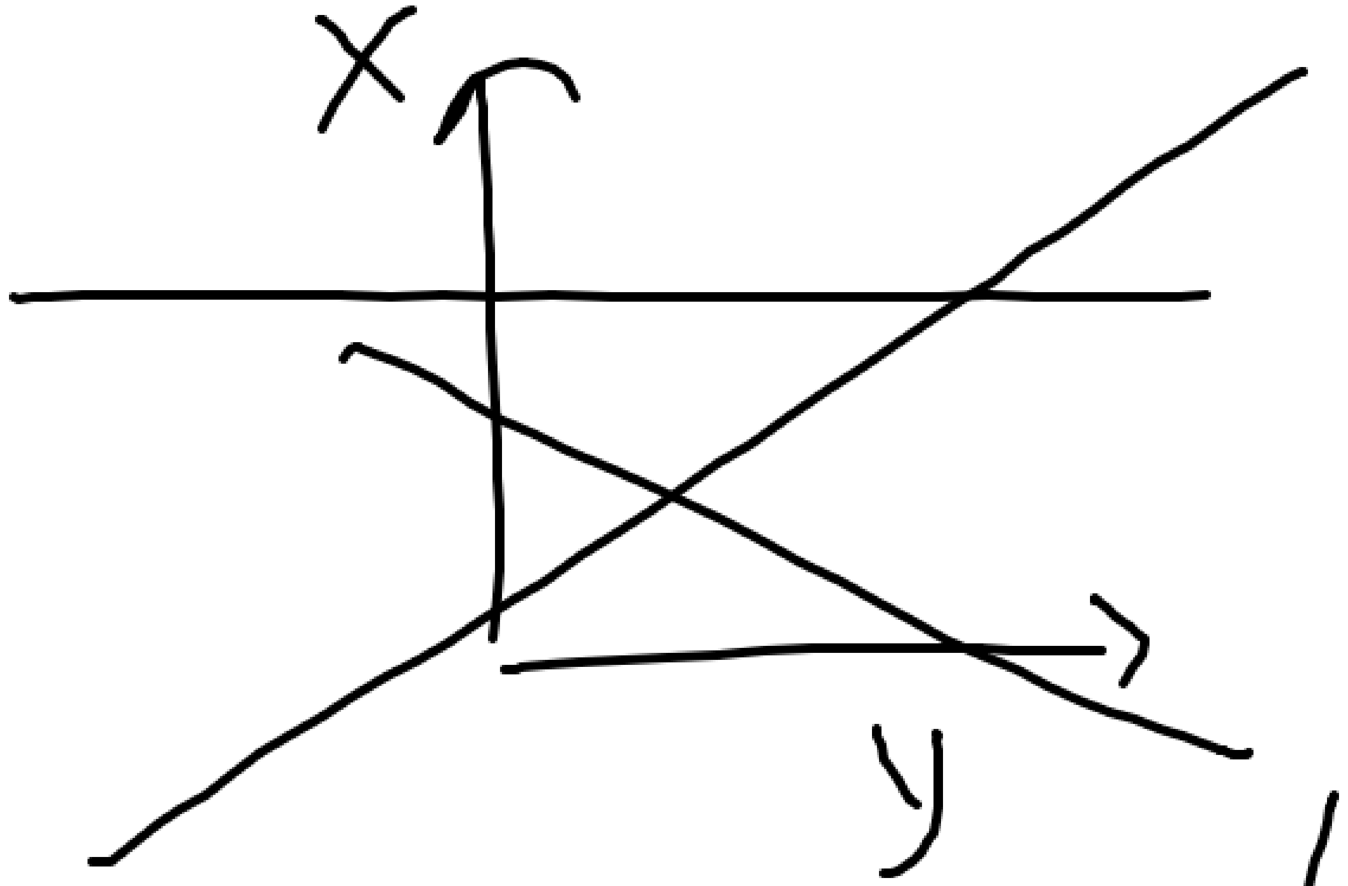

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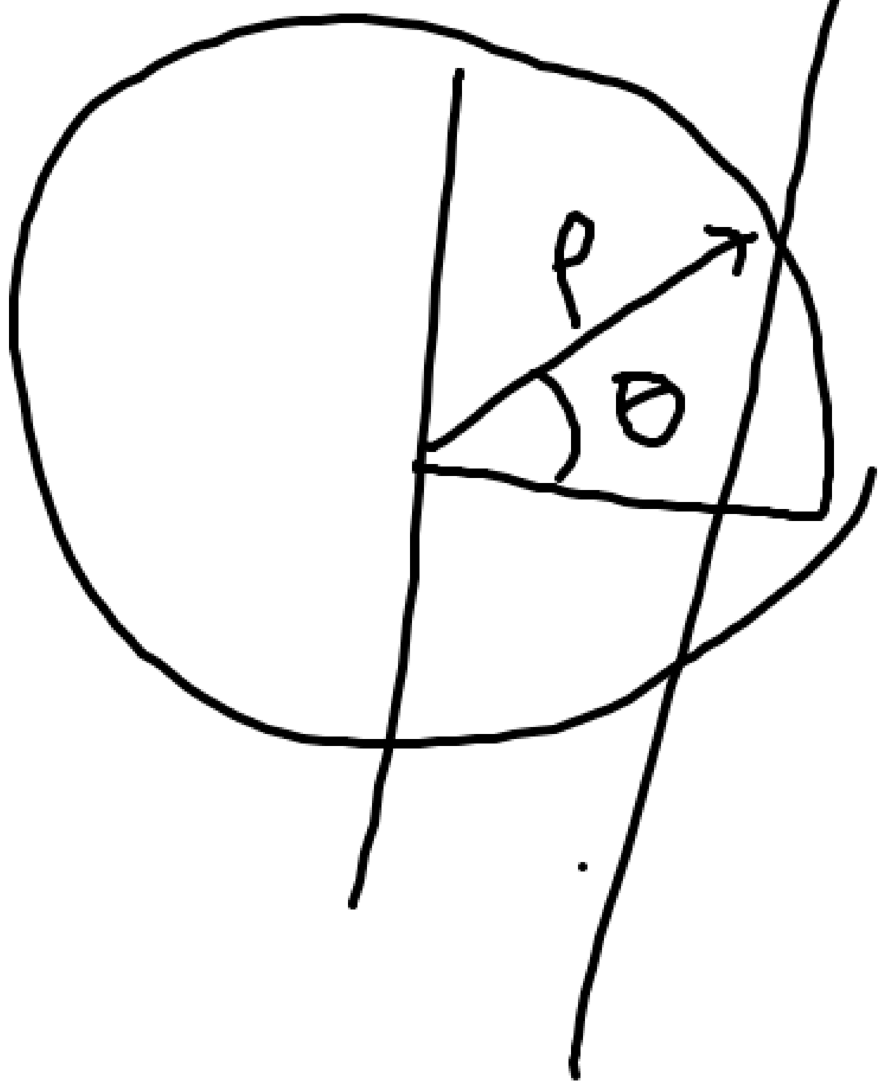

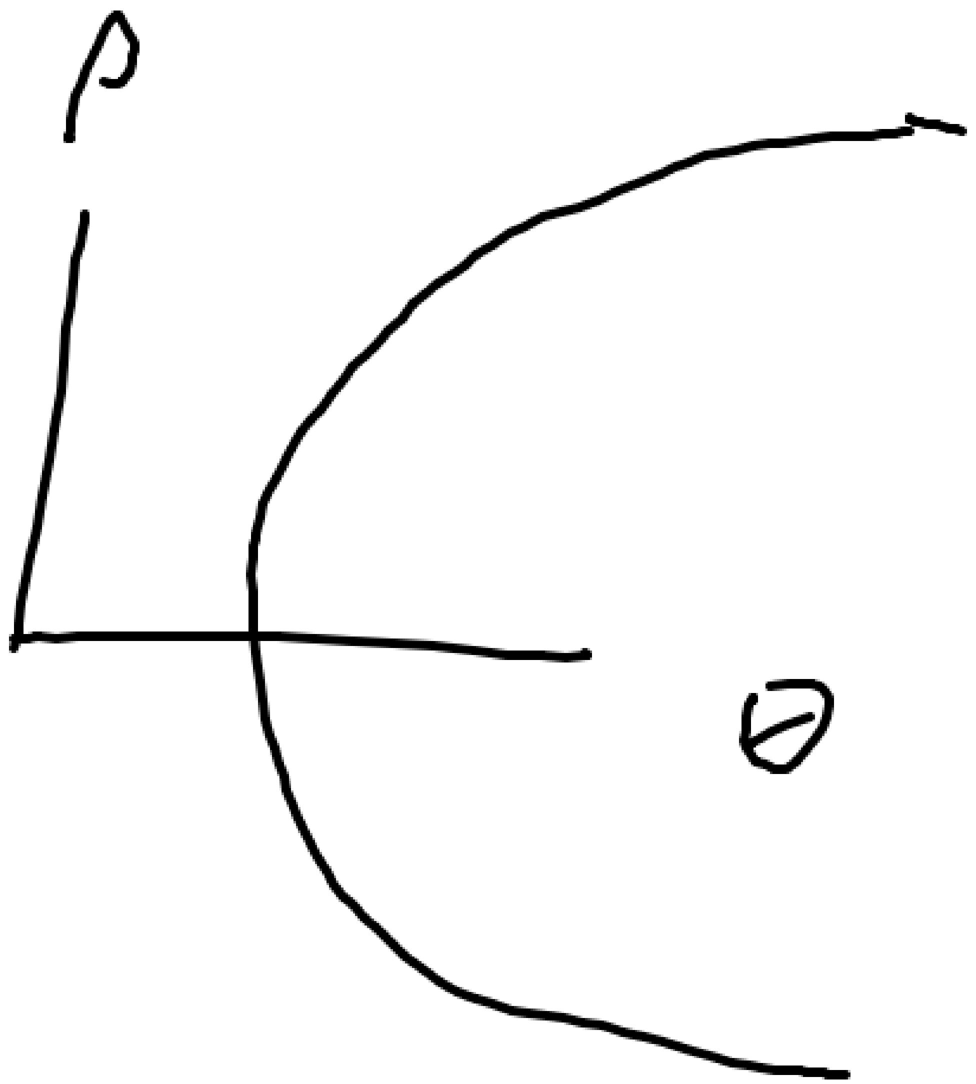

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$V^A W^B$ Fix V^A

g_{AB}

$(g_{AB} V^A)(W^B) = \#$

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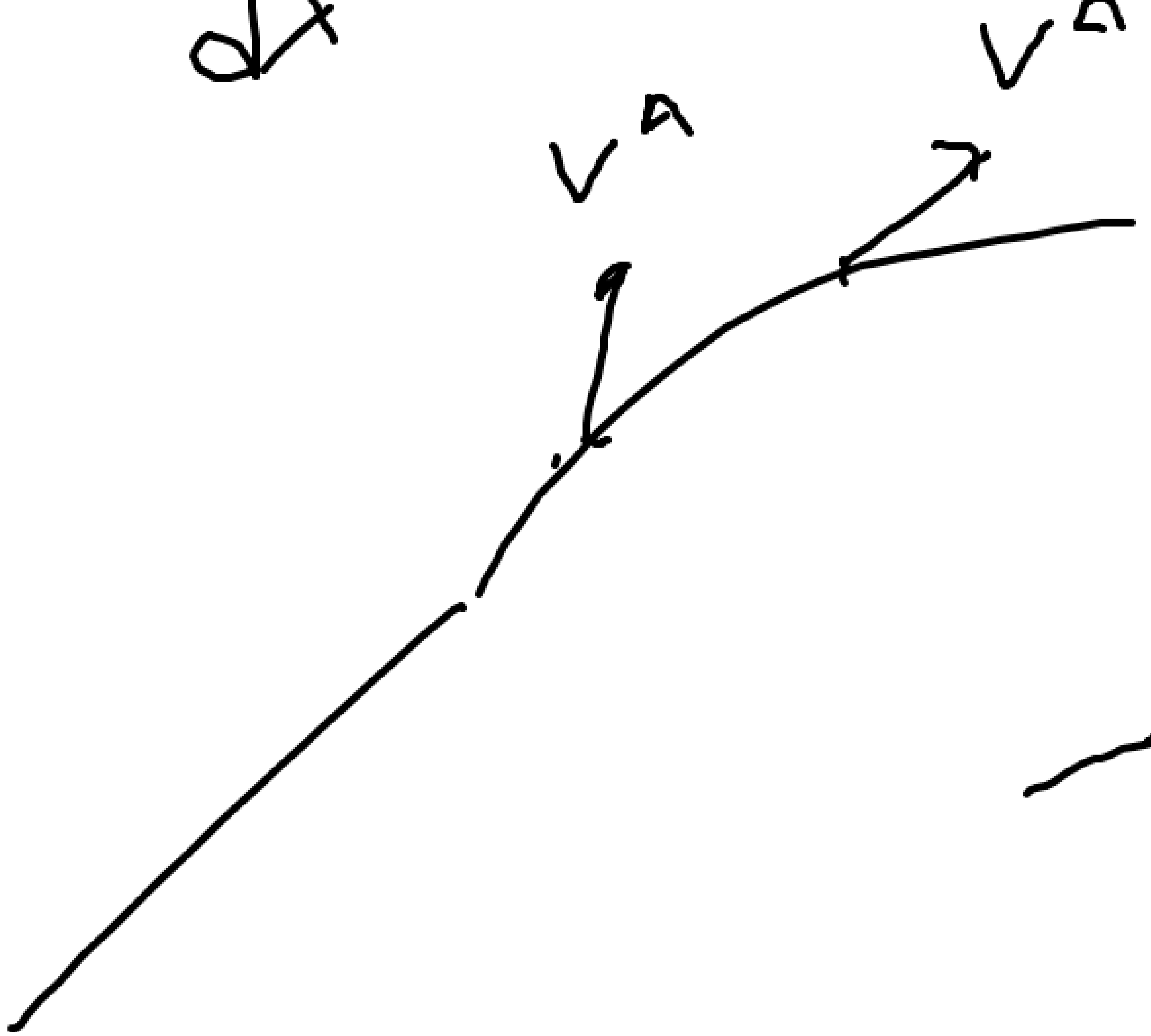

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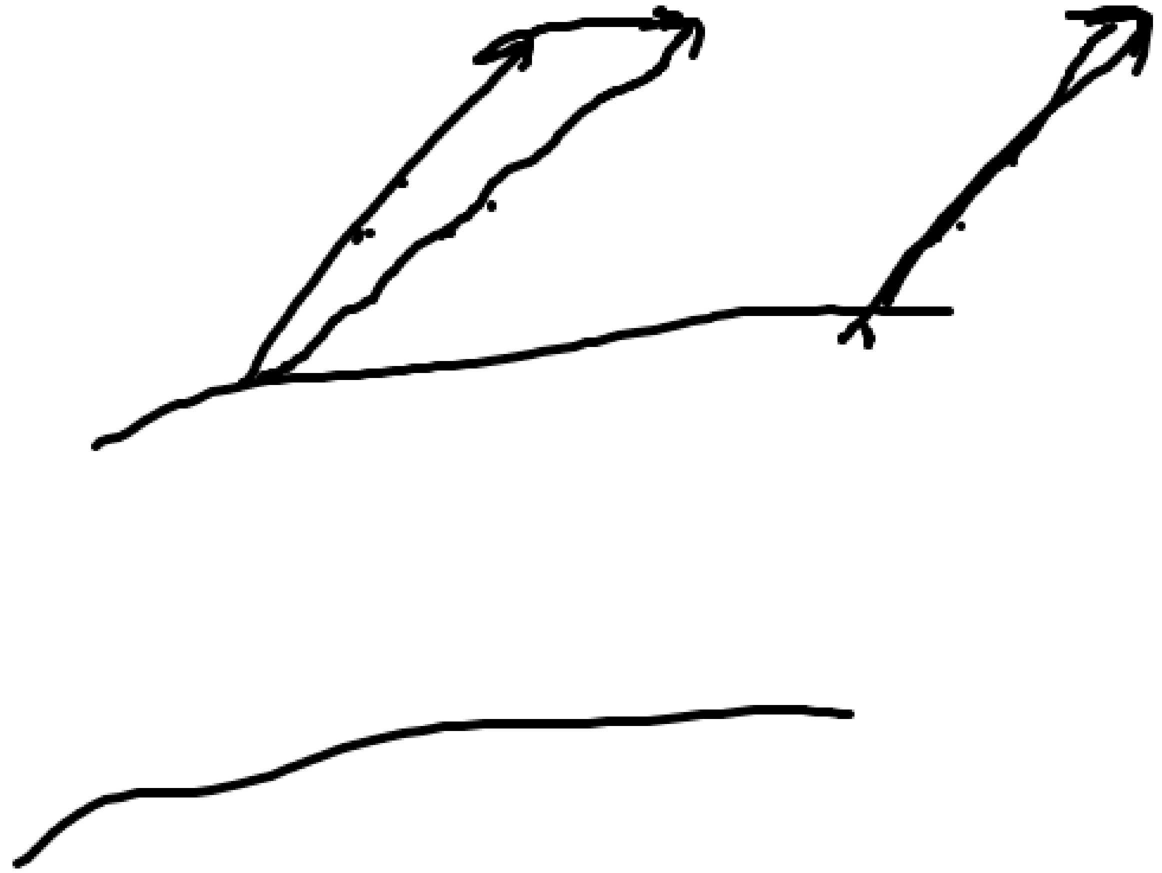
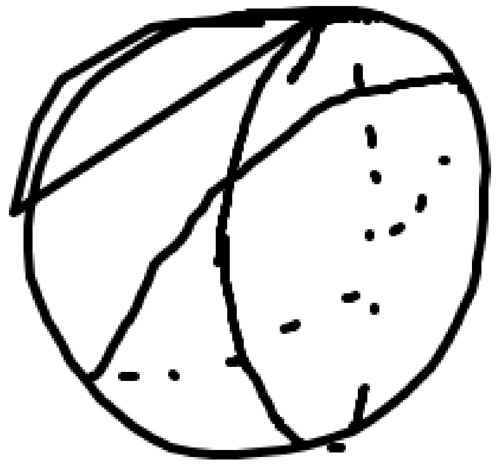
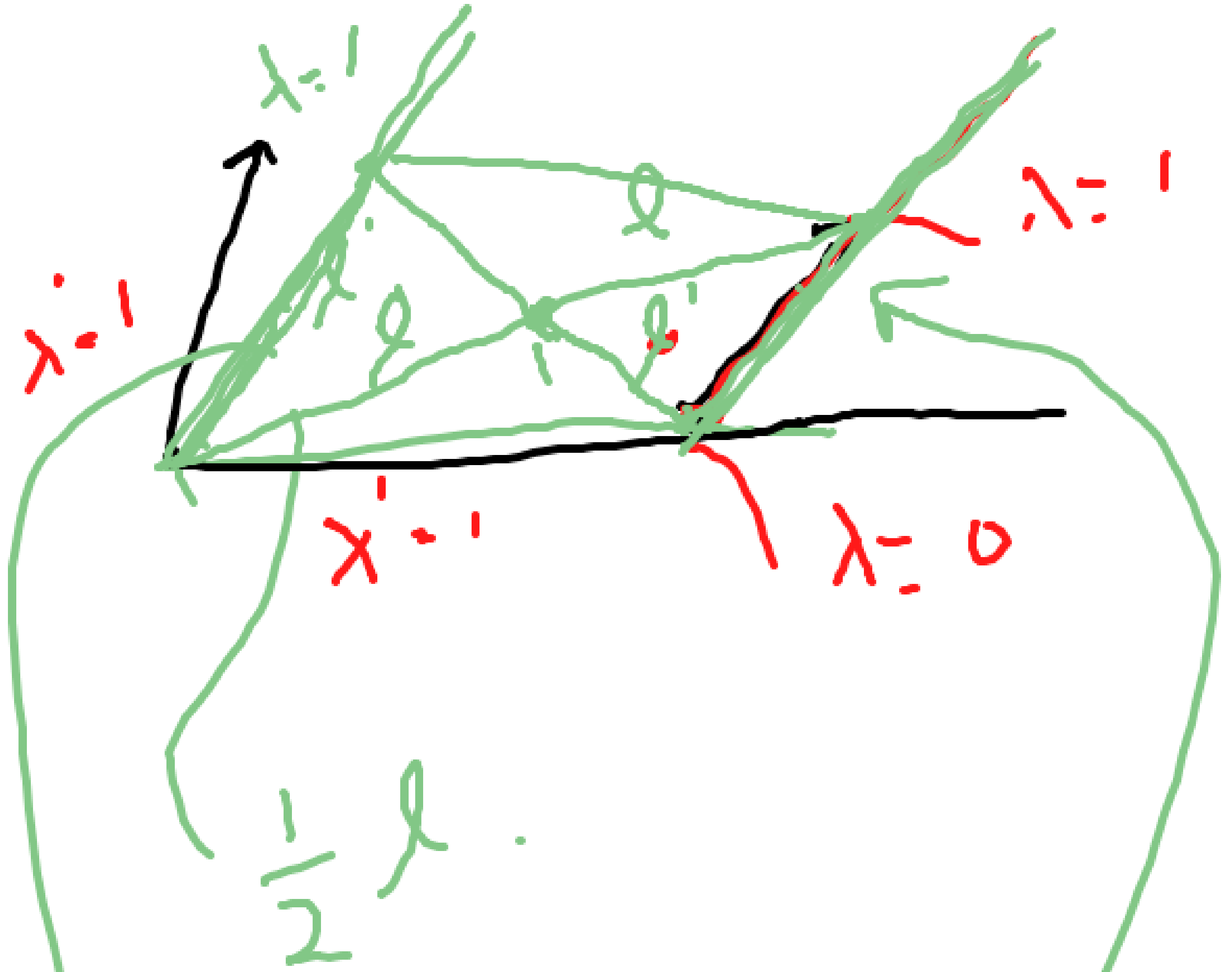
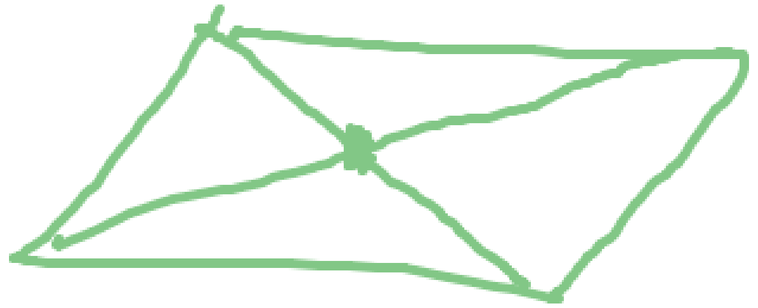
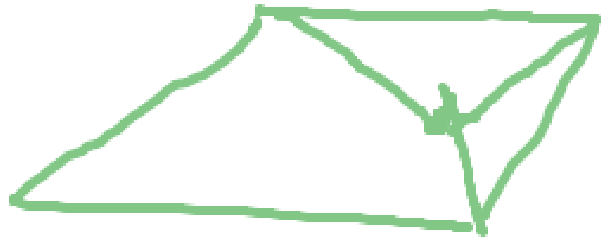






$$\lim_{\epsilon \to 0} \frac{P(V^A(\lambda+\epsilon)) - V^A(\lambda)}{\epsilon}$$





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