GR Assign 5

1) i) $R_{\mu\nu\rho\sigma}$ is antisymmetric in $\mu\nu$ and $\rho\sigma$
The components can only have two values 0, 1. Thus
R_{0101} is the only possible indep. component. Lets go to a coord system where $g_{00} = -g_{11} = 1$ and $g_{01} = 0$

Then $R_{00} = R_{0101}\, g^{11} = -R_{0101}$

and $R_{11} = R_{1010}\, g^{00} = R_{1010} = R_{0101}$

$= R_{0101}$

and $R_{01} = R_{0110}\, g^{10} = 0$

Thus $R_{\mu\nu} = -R_{0101}\, g_{\mu\nu}$

$R = -R_{0101}\, g_{\mu\nu} g^{\mu\nu} = -2R_{0101}$

ii) $g_{\mu\nu}$ embedded : $dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

2-D metric (r is const) : $r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

$${}^2\Gamma^{\phi}_{\theta\phi} = g^{\phi\phi}\, \tfrac{1}{2}\partial_\theta g_{\phi\phi} = \frac{\cos\theta}{\sin\theta} = {}^2\Gamma^{\phi}_{\phi\theta}$$

$${}^2\Gamma^{\theta}_{\phi\phi} = g^{\theta\theta}\, \tfrac{1}{2}(-\partial_\theta g_{\phi\phi}) = -\cos\theta\sin\theta$$

$$-R_{\phi\theta}{}^{\theta}{}_{\phi} = R_{\theta\phi}{}^{\theta}{}_{\phi} = (\partial_\theta \Gamma^{\theta}_{\phi\phi} - \partial_\phi \Gamma^{\theta}_{\phi\phi} - \Gamma^{\alpha}_{\phi\phi}\Gamma^{\theta}_{\alpha\phi}$$

$$= \sin^2\theta$$

GRAS

The extrinsic curvature is obtained by realizing that $\frac{dx}{d\lambda}$ the unit norm is $\nabla_\alpha r = (1,0,0)$

then $K_{ij} = \Gamma^r_{ij}$ $(ij = \theta, \phi)$

$= -\frac{1}{r} g_{ij}$ if in 1st case

and 0 in second case (since g_{ij} is indep of r.

1-c

iii) $ds^2 = -d\tau^2 + \tau^2 d\rho^2 + \tau^2 \sinh^2\rho \, d\phi^2$

The unit norm is $\nabla_\alpha \tau$ $(\alpha = 1, \beta_i = 0)$

Then $K_{ij} = \Gamma^\tau_{ij} = -\frac{1}{\tau} g_{ij}$

and $K_{\rho\rho} = \frac{1}{\tau} g_{\rho\rho}$ $K_{\phi\phi} = \frac{1}{\tau} g_{\phi\phi}$

$= \tau$ $= \tau \sinh^2\rho$

$K_{\rho\phi} = 0$.

${}^{(2)}R_{\rho\phi}{}^{\rho}{}_{\phi} = -\sinh^2(\rho)$

2 This needs to be considered in the linearized gravity limit (small mass.):

Special Relativity says the two energies must be conserved & the momentum must be conserved

so $m c^2 = E_1 + E_2 = \sqrt{p_1^2 + m_1^2} + \sqrt{p_2^2 + m_2^2}$

$p_1 = -p_2$

so p^2

$m^2 = 2p^2 + m_1^2 + m_2^2 + 2\sqrt{p^2 + m_1^2}\sqrt{p^2 + m_2^2}$

$\left[(m^2 - m_1^2 - m_2^2) + 2p^2\right]^2 = 4(p^2 + m_1^2)(p^2 + m_2^2)$

$p^2 = \dfrac{(m^2 - (m_1 + m_2)^2) + (m^2 - (m_1 - m_2)^2)}{4m^2}$

$\frac{p}{E_1} = v_1$

$-\frac{p}{E_2} = v_2$

$x_1 = \frac{p}{E_1} t \quad x_2 = -\frac{p}{E_2} t$

$Q_{xx} = m E_1 x_1^2 + m E_2 x_2^2$

$= p^2 \left(\frac{1}{E_1} + \frac{1}{E_2}\right) t^2$

$Q_{yy} = Q_{zz} = 0$

Trace free

$\bar{Q}_{xx} = \frac{2}{3} Q_{xx}$

$\bar{Q}_{yy} = -\frac{1}{3} Q_{yy}$

$\bar{Q}_{zz} = -\frac{1}{3} Q_{zz}$

$$\ddot{Q}_{xx} = 2\left(P^2\left(\frac{1}{E_1^2} + \frac{1}{E_2^2}\right)\right.\ \Theta(t)$$

$$\ddot{Q}_{yy} = \ddot{Q}_{zz} = 0$$

$$h^{TT}_{ij} = \left[\frac{\ddot{\bar{Q}}_{ij} - n^k n^l \ddot{\bar{Q}}_{kl}\, n_i n_j}{r}\right](t-r)$$

$$h^{TT}_{ij} = \frac{\ddot{Q}_{xx}}{r}\left[\delta_i^x \delta_j^x - n_x^2\, n_i n_j - \frac{1}{3}\left(\delta_{ij} - n_i n_j\right)\right]$$

$$n_i = \frac{\vec{r}_i}{|r|}$$

If $m_1 = m_2$, $E_1 = E_2$

$$\ddot{Q}_{xx} = \frac{4P^2}{m/2}\,\Theta(t-r) = \frac{4P^2}{m/2}\,\Theta(t-r)$$

$$= 8\left(\frac{m^2 - 4m_1^2}{4}\right) = 2\left(m^2 - 4m_1^2\right)\Theta(t-r)$$

The strength of h^{TT}_{ij} is constant at any r after the wave gets there (but before the particle arrives there).

3. Misprint GRAS 3-4

this metric is null dust, not a grav wave. Einsteins equs are again the linearized equs and $g_{uu}(u, x, y,)$ obeys the equ.

$$(\partial_u \partial_v - \partial_x^2 - \partial_y^2) g_{uv} = 8\pi\rho$$

$$\text{or } -4a\delta(u) = 8\pi\rho$$

Ie this is const density dust travelling along the null "sheet" $u = 0$.

The geodesic equs are

$$\frac{d^2u}{ds^2} = 0 \quad \text{or} \quad \frac{du}{ds} = \text{const}$$

Varn V

$$\frac{d^2x}{ds^2} + 2ax\delta(u)\left(\frac{du}{ds}\right)^2 = 0$$

$$\frac{d^2y}{ds^2} = ay\delta(u)\left(\frac{du}{ds}\right)^2 = 0$$

$$a\delta(u)(x^2+y^2)\left(\frac{du}{ds}\right)^2 + \frac{dv}{ds}\frac{du}{ds} - \left(\frac{dx}{ds}\right)^2 - \left(\frac{dy}{ds}\right)^2 = 0$$

But $\frac{du}{ds}$ is const $= \dot{u}_0$

Thus $\frac{d}{ds} = \dot{u}_0 \frac{d}{du}$

Thus $\frac{dx}{du} = ax_0\theta(u) \quad \frac{dy}{du} = ay_0\theta(u)$

and $X = X_0(1 + a\theta(u)u)$

$y = y_0(1 + a\theta(u)u)$

$V = V_0 - \theta(u)a(x_0^2 + y_0^2) + u\frac{u}{i_0}$

$\frac{dV}{ds} i_0 = 1$ is required where $u \neq 0$.

I.e. there is a step function discont. in V at $u = 0$.

Note that if $a < 0$ (which one needs to have the energy density > 0) then x, y go to zero when $u = -1/a$ — focusing. This is indep. of x_0, y_0.

Ds in lecture.

If $ds^2 = d\tilde{v}\,du - f^2(u)\,d\tilde{x}^2 - g^2(u)\,d\tilde{y}^2$

we can make these ~~dep~~ like the above by

$x = f(u)\tilde{x} \quad y = g(u)\tilde{y}$

to give

$$ds^2 = \left(\left(\frac{\dot{f}}{f}\right)^2 x^2 + \left(\frac{\dot{g}}{g}\right) y^2\right) du^2 + du\left(d\tilde{v} - 2\frac{\dot{g}}{g} y\, dy - 2\frac{\dot{f}}{f} x\, dx\right) - dx^2 - dy^2$$

Taking $\tilde{V} = \left(V - \frac{\dot{g}}{g} y^2 - \frac{\dot{f}}{f} x^2\right)$

we get $ds^2 = \left(\frac{\ddot{f}}{f} x^2 + \frac{\ddot{g}}{g} y^2\right) du^2 + du\,dV - dx^2 - dy^2$

thus $\frac{\ddot{f}}{f} = \frac{\ddot{g}}{g} = a\,\delta(u)$

or $\ddot{f} = a\,\delta(u)\,f$

$\partial_u f(0^+) - \partial_u f(0^-) = a\,f(0)$ at $u=0$

and $\ddot{f} = 0$ for $f \neq 0$.

$\dot{f}(a) = 1$ for $u<0$

$f = 1 + a\,u\,(\theta(u))$

Now, geodesic eqns are again

$\frac{d^2u}{ds^2} = 0 \qquad \frac{d}{du} f^2(u) \frac{dx}{du} = \frac{d}{du} f^2 \frac{dy}{du} = 0$

or $x = x_0\left(1 + \frac{1}{f^2(u)}\right) \to \infty$

at $u = -\frac{1}{a}$ I.e singularity.

Note this is true if null or timelike geodesic. I.e, all incoming dust focused at $u = -\frac{1}{a}$.

A student could assume (from my comment about grav wave) that I made a misprint and I meant a $\delta(u)(x^2-y^2)$. In that case one finds $f(u) = (1+au\theta(u))$

$g(u) = (1-au\theta(u))$

or $X = X_0(1+au\theta(u))$

$y = y_0(1-au\theta(u))$

and one converges and the other diverges depending on the sign of a. The converging one produces a singularity, while the diverging one does not. However even if x_0 or x say ($a<0$) goes to 0, the metric is non singular since it is just $dv^2 - dx^2 - dy^2$ for $u \neq 0$. at least if a is small enough. Because V is discont. at $u=0$, ($V = V_0 \pm (x_0^2 - y_0^2)a$) for large enough x_0, y_0 can get overlaps in V). Would need more analysis.