Physics 501-22
Assignment 3

1.) Hardy system: Given the state

$$|\Psi\rangle = \alpha\,|\uparrow\rangle\,|1\rangle + \beta\,|\downarrow\rangle\,(S\,|1\rangle + C\,|0\rangle) \tag{1}$$

where this is a unit vector with $|\alpha|^2 + |\beta|^2 = 1$ and $|C|^2 + |S|^2 = 1$

i)Argue that we can always choose the coefficients as real and positive by adjusting the phases of the basis vectors..

We have three possible phases. One phase, the overall phase, is irrelevant, since we are only interested in the state up to a phase. The other two phases can be absorbed into the relative phase of $|1\rangle$ and $|0\rangle$. The other phase can be absorbed into $|\uparrow\rangle$ vs $|\downarrow\rangle$

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ii) Find the value of S that minimizes the probability of having the final value of "D" be equal to +1. (Recall from the lectures that one has two systems, with A,B being attributes of the first system, and C,D of the second. They are such that $A \to 1 \Rightarrow C \to 1, C \to 2 \Rightarrow B \to 1, B \to 2 \Rightarrow D \to 1$, but $A \to 1$ does not imply that $D \to 1$ (in fact the probability that when A has value 1 it is highly improbably that D also has the value 1. $A \to 1$ here means A is found to have value 1. $\Rightarrow$ means "implies that"− ie if one makes measurements on the system, then it is always true that if A and X are measured, then whenever A is found to value 1, X always also has value 1.)

Oh dear, I have overloaded the definiton of C. It is both the attribute of the second system and a coefficient of the wavefunction. Since $C^2 + S^2 = 1$ in the second, I will always use $\sqrt{1-S^2}$ for the coefficient.

Define $\alpha = s$,$\beta = \sqrt{1-s^2}$, since $\alpha^2 + \beta^2 = 1$ Then

$$|A\rangle = |\uparrow\rangle \tag{2}$$

$$|C\rangle = \frac{\langle A|\,|\psi\rangle}{|\,\langle A|\,|\psi\rangle\,|} = |1\rangle \tag{3}$$

$$\tag{4}$$

where $|\,\langle A|\,|\psi\rangle\,| = \sqrt{(\langle\psi|\,|A\rangle)(\langle A|\,|\psi\rangle)}$.

$$|B\rangle = \frac{\langle C|\,|\psi\rangle}{|\,\langle C|\,|\psi\rangle\,|} \tag{5}$$

$$= \frac{s\,|\uparrow\rangle + \sqrt{1-s^2}S\,|\downarrow\rangle}{\sqrt{s^2 + (1-s^2)S^2}} \tag{6}$$

$$|D\rangle = \langle B|\,|\psi\rangle = \frac{s^2\,|1\rangle + (1-s^2)S^2\,|1\rangle + (1-s^2)S\sqrt{1-S^2}\,|0\rangle}{\sqrt{(s^2 + (1-s^2)S^2)^2 + (1-s^2)^2S^2(1-S^2))}} \tag{7}$$

$$= \frac{(s^2 + (1-s^2)S^2)\,|1\rangle + (1-s^2)S\sqrt{(1-S^2)}\,|0\rangle}{\sqrt{(s^2 + (1-s^2)S^2)^2 + (1-s^2)^2S^2(1-S^2)}} \tag{8}$$

Then the probability of D given A, is the probability of D and A over the probability of A or

$$\mathcal{P}_{D|A} = \frac{\mathcal{P}_{D\&A}}{\mathcal{P}_A} = \frac{\langle A| \langle D| |\psi\rangle |^2}{| \langle A| |\psi\rangle |^2} \tag{9}$$

$$= (\langle C| |D\rangle)^2 = \frac{(s^2 + (1-s^2)S^2)^2}{(s^2 + (1-s^2)S^2)^2 + (1-s^2)^2 S^2(1-S^2)} \tag{10}$$

To find the minimum over S, the easiest way is to let $S^2 = z$ and take the derivative with respect to z. This gives

$$-\frac{(s-1)^2 * ((z-1) * s^2 - z) * ((z-1) * s^2 + z) * (s+1)^2}{((z-1) * s^4 - z)^2} = 0 \tag{11}$$

Since $0 < z < 1$, the only solution is $z = \frac{s^2}{1+s^2}$ which gives

$$\mathcal{P}_{D|A} = \frac{4s^2}{(1+s^2)^2} \tag{12}$$

which goes from 0 for s=0 to 1 for s=1. Since for $s = 0$, the probability of measuring A goes to 0 as well, we need s to be small, but not zero.

Note that if $S^2 = 0$ or $S^2 = 1$ for non-zero s, the probability of D given A, $\mathcal{P}_{D|A}$ is unity. Since that is the maximum value for the probability, the extremum above must be a minimum for constant s..

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iii) Given that value of S, what is the largest value of the the ratio of the eigenvalues $\lambda_1,\ \lambda_2$ where the two λ are the two eigenvalues of the reduced density matrix of particle 1 with $\lambda 1$ being the smallest of the eigenvalues.

If $S^2 = \frac{s^2}{1+s^2}$, we have

$$|\psi\rangle = s|\uparrow\rangle\, 1 + \sqrt{\frac{1-s^2}{1+s^2}}\, |\uparrow\rangle\, (s\,|1\rangle + |0\rangle) \tag{13}$$

Tracing out over the first system we get

$$\rho_2 = s^2\, |1\rangle\, \langle 1| + \frac{1-s^2}{1+s^2}(s\,|1\rangle + |0\rangle)(s\,\langle 1| + \langle 0|) \tag{14}$$

$$= 2\frac{s^2}{1+s^2}\, |1\rangle\, \langle 1| + s\frac{1-s^2}{1+s^2}(|1\rangle\, \langle 0| + |0\rangle\, \langle 1|) + \frac{1-s^2}{1+s^2}(|0\rangle\, \langle 0|) \tag{15}$$

The trace of this $\frac{(2s^2+(1-s^2))}{1+s^2}$ is unity. The determinant

$$det = 2\frac{s^2}{1+s^2}\frac{1-s^2}{1+s^2} - (s\frac{1-s^2}{1+s^2})^2 = s^2\frac{1-s^2}{1+s^2} \tag{16}$$

Thus the eigenvalue equation is

$$\lambda^2 - \lambda + s^2 \frac{1-s^2}{1+s^2} = 0 \tag{17}$$

$$\lambda = \frac{1}{2}(1 \pm \sqrt{1 - 4\frac{s^2(1-s^2)}{1+s^2}} \tag{18}$$

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(recall that the density matrix for the second particle associated with a pure entangled state on the whole system is

$$|\Psi\rangle = \sum_i \lambda_i |\phi_i\rangle |\psi_i\rangle \tag{19}$$

is

$$\rho = \sum_{i,j} \lambda_i^* \lambda_j \langle\phi_i| |\phi_j\rangle |\psi_j\rangle \langle\psi_i| \tag{20}$$

where $|\phi\rangle$ is a state for the first particle/system, while $|\psi\rangle$ is a state for the second particle/system. For the Hardy system, use the two component vector to find the matrix representing the reduced density matrix for the second particle.

2) Assume that we have a Hamiltonian

$$H = \frac{1}{2}\left(\frac{p_1^2}{m_1^2} + \frac{p_2^2}{m_2} + k_1 x_1^2 + k_2 x_2^2 + 2\epsilon x_1 x_2\right) \tag{21}$$

a)What are the 4 eigenvalues $\pm i\omega_1, \ \pm\omega 2$ of the Hamiltonian equations for this Hamiltonian in terms of the constants $m_1, m_2, k_1, k_2, \epsilon$.

The eigenvalues are given from the equations of motion by assuming that all dynamic variables have their derivative equal to $-i\omega$ times themselves. Thus we have

$$\omega x_1 = \frac{p_1}{m_1} \tag{22}$$

$$\omega x_2 = \frac{p_2}{m_2} \tag{23}$$

$$\omega p_1 = -(k_1 x_1 + \epsilon x_2) \tag{24}$$

$$\omega p_2 = -(k_2 x_2 + \epsilon x_1) \tag{25}$$

From the first and third, and the 2nd and 4th, we get

$$\omega^2 x_1 = -\frac{(k_1 x_1 + \epsilon x_2)}{m_1} \tag{26}$$

$$\omega^2 x_2 = -\frac{(k_2 x_2 + \epsilon x_1)}{m_2} \tag{27}$$

If we define $\Omega_1^2 = k_1/m_1$ and $\Omega_2^2 = k_2/m^2$ we find

$$(\omega^2 - \Omega_1^2)(\omega^2 - \Omega_2^2) = \frac{\epsilon^2}{m_1 m^2} \tag{28}$$

or

$$\omega^2 = \frac{1}{2}\left((\Omega_1^2 + \Omega_2^2) \pm \sqrt{(\Omega_1^2 - \Omega_2^2)^2 + 4\frac{\epsilon^2}{m_1^2 m_2^2}} \right) \tag{29}$$

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b) Is there any condition on k_i, m_i, ϵ such that $\omega_1 = \omega 2$?

Only if $\epsilon = 0$

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c) If $m_1 = m_2$, $k_1 = k_2$, is there any condition on ϵ such that the eigenvalues are not purely imaginary?

From part a, if $\Omega 1 = \Omega 2$ then

$$\omega^2 = \Omega^2 \pm \frac{\epsilon}{m} \tag{30}$$

thus if $\epsilon > m\Omega^2$, two of the solutions for ω have imaginary ω.

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d) What are the normalised (using the symplectic norm) eigenvectors if $m_1 = m_2$, $k_1 = k_2$ and $\epsilon \neq 0$?

$$< \{x_1, x_2\}, x_1, x_2 >= i[(x_1^* p_1 + x_2^* p_2) - (p_1^* x_1 + p_2^* x_2)] = 2\omega(x_1^2 + x_2^2) \tag{31}$$

where

$$\omega^2 = \Omega^2 \pm \frac{\epsilon}{m^2} \tag{32}$$

Actually, interchange symmetry of this Hamiltonian, ($x_1 \leftrightarrow x_2$) the interchange symmetry is a a symmetry of the solutions. Ie, defining

$$y_s = (x_1 + x_2)/\sqrt{2} \quad ; \quad p_s = (p_1 + p_2)/\sqrt{2} \tag{33}$$
$$y_a = (x_1 - x_2)/\sqrt{2} \quad ; \quad p_a = (p_1 - p_2)\sqrt{2} \tag{34}$$
$$\tag{35}$$

(which is a cannonical transformation since

$$p_1 \dot{x}_1 + p_2 \dot{x}_2 = p_s \dot{y}_s + p_a \dot{y}_a \tag{36}$$

and

$$H = \frac{1}{2}((p_a^2 + p_s^2)/m + (\Omega^2 + \epsilon)x_s^2 + (\Omega^2 - \epsilon)x_a^2) \quad (37)$$

$$= \quad (38)$$

Ie, these modes are decoupled (no coupling interaction).

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3). Consider the Hamiltonian $H = \frac{1}{2}(p^2 - x^2)$.

a)What are the eigenvalues of the Hamiltonian (The "diagonalization of the Hamiltonian" values for omega? Show that there are no purely real eigenvalues.

$$-i\omega x = p \quad ; \quad -i\omega p = x \quad (39)$$

or $\omega^2 = -1$. Ie, the eigenvalues are purely imaginary. This implies that the the solutions are $e^{-i\omega t}$ and $e^{i\omega t}$ are both real.– $e^{\pm t}$.

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b)Find a positive norm, normalised mode. (Recall that if you have two independent classical solution, the sum of the first plus i times the second is a complex mode solution.) What is the time dependence of this mode. Show that its norm is independent of time explicitly.

To get a complex solution one has to take a complex sum of these two modes, eg

$$x = \alpha(e^t + ie^{-t}) \quad (40)$$

with α real (one can take arbitrary combinations, eg, $\alpha e^t + \beta e^{-t}$ with arbitrary complex α and β, as long as β/α is not real. Now its complex conjugate is another solution. Then

$$p = \partial_t x = \alpha(e^t - ie^{-t}) \quad (41)$$

and the norm is

$$< x, x >= i|\alpha|^2(e^t - ie^{-t})(e^t - ie^{-t} - (e^t - ie^{-t})(e^t - ie^{-t})) \quad (42)$$

$$= 4 \quad (43)$$

Thus to normalise this, we need to take $\alpha = 1/2$. Note that this is constant, even though the modes are either exponentially growing of dying.

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d)Find the Annihilation and Creation operators corresponding to this mode, and show explicitly that they are independent of time.

The Heisenberg solutions for the equations of motion are

$$\partial_t X = P \quad (44)$$

$$\partial_t = X \quad (45)$$

with solutions

$$X = X_0 cosh(t) + P_0 sinh(t) \tag{46}$$
$$P = P_0 cosh(t) + X_0 sinh(t) \tag{47}$$

Then we have

$$A = < x, X > \tag{48}$$
$$= i\frac{1}{2}((e^t - ie^{-t})(cosh(t)P_0 + \sinh(t)X_0) - (e^t + ie^{-t})(X_0 cosh(t) + P_0 \sinh(t)) \tag{49}$$
$$= \frac{i}{4}\left[\left((e^{2t}(X_0 + P_0) + (P_0 - X_0) - i(P_0 + X_0) - ie^{-2t}(P_0 - X_0))\right)\right. \tag{50}$$
$$\left. - \left(e^{2t}(X_0 + P_0) + (X_0 - P_0) + i(X_0 - P_0) + ie^{-2t}(X_0 - P_0)\right)\right] \tag{51}$$
$$= \frac{1}{2}(P_0 + X_0 + i(P_0 - X_0)) \tag{52}$$

which is independent of t.
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e) What is the quantum Hamiltonian in terms of these annihilation and creation operators?

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The Hamiltonian must be Hermitian and is time independent. Thus we must have

$$H = \alpha A^2 + \alpha^* A^{\dagger 2} + \beta(A^\dagger A) + \gamma \tag{53}$$

where α is complex, β and γ are real. Now, we have

$$A^2 = \frac{1}{2}(P_0 X_0 + X_0 P_0 + I * (P_0^2 - X_0^2)) \tag{54}$$

and

$$H = \frac{1}{2}(P^2 - X^2) = \frac{1}{2}(P_0^2 - X_0^2) \tag{55}$$

Thus

$$H = -\frac{i}{2}(A^2 - (A^2)^\dagger) = -\frac{i}{2}((A^2 - A^{\dagger 2})) \tag{56}$$

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