Physics 501-20
Assignment 1

1.) Consider the Hamiltonian

$$H = \frac{1}{2}(P^2 + (1 + \epsilon\delta(t))Q^2) \tag{1}$$

Assume that at $t = 0^-$ (ie just before $t = 0$ the operators P, Q are given by $P = P_0, \quad Q = Q_0$.

Assume that the intial state of this quantum system is the lowest energy state at time $t = 0^-$ (the ground state).

a) Find the Heisenberg equations of motion, and solve them for arbitrary time in terms of $P_0 = P(0^-)$ and $Q_0 = Q(0^-)$

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Assume $\hbar = 1$

$$\partial_t Q = -i[Q, H] = -\frac{i}{2m}[Q, P^2] = P \tag{2}$$

$$\partial_t P = -i[P, H] = -(1 + \epsilon\delta(t))Q \tag{3}$$

For both $t \neq 0$ this is just the free Harmonic oscillator For t¡0, the solution is

$$Q = Q_0 cos(t) + P_0 \frac{sin(t)}{m} \tag{4}$$

$$P = P_0 cos(t) - Q_0 \sin(t) m \tag{5}$$

$$\tag{6}$$

$t > 0$

$$Q = Q(0^+) cos(t) + P(0^+) sin(t) \tag{7}$$

$$P = P(0+) cos(t) - Q(0+) \sin(t) \tag{8}$$

Where 0^+ is the time just after t=0. Integrating from $-\mu$ to $+\mu$ where $\mu \to 0$, we have

$$Q(0^+) - (Q_0) = 0 \tag{9}$$

$$P(0^+) - P_0 = -\epsilon Q_0 \tag{10}$$

The minimum energy state is (where q are the eigenvalues of Q_0)

$$\psi(q) = \frac{1}{(2\pi)^{1/4}} e^{-\frac{q^2}{2}} \tag{11}$$

which obeys

$$(\partial_q + q)\psi(q) = 0 \tag{12}$$

Thus after $t = 0$, we have

$$Q(t) = Q(0+)\cos(t) + P(0+)\sin(t) = Q_0\cos(t) + (P_0 - \epsilon Q_0)\sin(t) \quad (13)$$
$$P(t) = P(0+)cos(t) - Q(0+)\sin(t) = (P_0 - \epsilon Q_0)cos(t) - Q_0 sin(t) \quad (14)$$
$$(15)$$

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b)Define the Annihilation operators $A_0 = \frac{1}{\sqrt{2}}(Q_0 + iP_0)$. Show that $[A_0, A_0^\dagger] = 1$ and that $H(t < 0) = \frac{1}{2}A_0 A_0^\dagger + A_0^\dagger A_0$. The minimum energy state $|0\rangle$ will therefor be given by

$$A_0 |0\rangle = 0 \quad (16)$$

Given the solution of the Heisenberg equations of motion, write the solutions for $P(t), Q(t)$ in terms of $A, \quad A^\dagger$. at all times.
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If $A = \frac{1}{\sqrt{2}}(Q_0 + iP_0)$ then

$$Q_0 = \frac{1}{\sqrt{2}}(A + A^\dagger)) \quad (17)$$
$$P_0 = -\frac{i}{\sqrt{2}}(A - A^\dagger) \quad (18)$$

Then $A|0\rangle = 0$ becomes $(Q_0 + iP_0)|0\rangle = 0$ or writing this in the Q_0 basis $\partial_q\psi(q) + q\psi(q) = 0$ whicha has as normlized soution the funtion given in part a.

Before t=0, the solution is

$$Q(t) = \frac{1}{\sqrt{2}}(A_0 e^{-it} + A_0^\dagger e^{it}) \quad (19)$$
$$P(t) = -\frac{i}{\sqrt{2}}(A_0 e^{-it} - A_0^\dagger e^{it} \quad (20)$$

For t¿0, we have

$$Q(t) = Q_0 cos(t) + (P_0 - epsilon Q_0)sin(t) = \frac{1}{\sqrt{2}}\left[A_0 e^{-it} + A_0^\dagger e^{it} - \epsilon(A_0 + A_0^\dagger)sin(t)\right] \quad (21)$$
$$P(t) = \frac{1}{\sqrt{2}}(-i(A_0 e^{-it} - A_0^\dagger e^{it}) - \epsilon(A_0 + A_0^\dagger)cos(t)) \quad (22)$$

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c) Find the expectation value of the energy H in the state $|0\rangle$ as a funtion of time.
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$$H = \frac{1}{2}(P(t)^2 + Q(t^2)) = \frac{1}{2}(P_0^2 + Q_0^2) = \frac{1}{2}(A^\dagger A + AA^\dagger) \quad (23)$$

Since the ground state is $A|0\rangle = 0$, we have

$$\langle H \rangle = \frac{1}{2} \langle A^\dagger A + [A, A^\dagger] \rangle = \frac{1}{2} \tag{24}$$

while for t¿0, again H is conserved and it value will be the same as at $t = 0^+$

$$H = \frac{1}{2}(P(0^+)^2 + Q(0^+)^2) = \frac{1}{2}(P_0^2 + Q_0^2 - \epsilon(P_0 Q_0 + Q_0 P_0) + \epsilon^2 Q_0^2) \quad (2$$

$$= \frac{1}{2}\{(A_0 A_0^\dagger + A_0^\dagger A_0) + i\frac{1}{2}\epsilon((A_0 - A_0^\dagger)(A_0 + A_0^\dagger) + (A_0 + A_0^\dagger)(A_0 - A_0^\dagger)) + \frac{1}{2}\epsilon^2(A_0^2 + A_0^{\dagger 2} + (A_0 A_0^\dagger + A_0^\dagger A_0)\} \quad (2$$

$$= \frac{1}{2}[(A_0 A_0^\dagger + A_0^\dagger A_0) i\epsilon (A_0^2 - A_0^{\dagger\ 2}) + \frac{1}{2}\epsilon^2 (A_0 + A_0^\dagger)^2 \quad (2$$

Taking the expectation value on the state $|0\rangle$ and recalling that $A_0|0\rangle = \rangle 0| A^\dagger = 0$ we have

$$\langle\langle\rangle\, H\rangle = \frac{1}{2}(1 + \frac{1}{2}\epsilon^2) \tag{28}$$

(using the relations $A_0|0\rangle = 0 = \rangle 0| A_0^\dagger$).

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d)Solve the Schroedinger equation for this problem with the same initial conditions, and explicitly find the expectation value of the energy as a function of time. (If necessary, solve this to lowest non-trivial order in ϵ. Note that the Heisenberg equations can be solved exactly to all orders in ϵ).

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The Schroedinger equation is

$$\partial_t \psi(x,t) = -iH|\psi\rangle = -i(-\frac{1}{2}\partial_x^2 \psi(x,t) + \frac{1}{2}x^2 \psi(x,t) + \epsilon\delta(t)x^2 \tag{29}$$

The ground state has energy 1/2 and is Gaussian in x or

$$\psi(x,t) = N e^{-it/2} e^{\frac{-x^2}{2}} \tag{30}$$

. At t=0 we have to get the solution past the delta function. This is not trivial, but there is a trick. At the delta function, the delta function dominates the evolution, so we can write the equation in the immediate vicinity of t=0 as

$$\partial_t \psi(x,t) = -i\epsilon x^2/2\delta(t)\psi(x,t) \tag{31}$$

dividing by $\psi(x,t)$ we have

$$\partial_t ln(\psi(t,x)) = -i\epsilon x^2/2\delta(t) \tag{32}$$

or

$$ln(\psi(x,0^+) - ln(\psi(x,0^-)) = -i\epsilon x^2/2 \tag{33}$$

or

$$\psi(x,0+) = Ne^{-(1-i\epsilon)x^2/2} \tag{34}$$

Note that this extra term is an x dependent phase, and so it does not alter the normalization, N.

This is not an energy eigenstate, so after $t = 0$ the solution in time is very complex. However, since the energy is conserved, we can calculate the expectation value of the energy at time $t = 1 + \mu$ which will be conserved for all times after the interaction. This of course gives us the same answer as for the Heisenberg representation.

The state is given by

$$(iP_0 + Q_0)\psi = 0 \tag{35}$$

Solving for $Q_0,\ P_0$ in terms of $P(t),\ Q(t)$ we get

$$P_0 = P(t)(cos(t) - \epsilon sin(t)) + Q(t)(sin(t) + \epsilon cos(t)) \tag{36}$$

$$Q_0 = Q(t)(cos(t) - P(t)sin(t)) \tag{37}$$

and

$$iP_0 + Q_0 = iP(t)(e^{-it} - \epsilon sin(t)) + Q(t)(e^{-it} + i\epsilon cos(t)) \tag{38}$$

defining q as the eigenvalues of $Q(t)$, and thus $iP(t)f(q) = \partial_q f(q)$ we have,

$$(iP_0 + Q_0)\psi(q) = 0 \tag{39}$$

$$\psi(t,q) = N^+ e^{-\frac{q^2}{2}\frac{e^{-it}+i\epsilon cos(t)}{e^{-it}-i\epsilon sin(t)}} \tag{40}$$

At $t = 0^+$, this is

$$\psi(0^+,q) = N^+ e^{-\frac{q^2}{2}(1+i\epsilon)} \tag{41}$$

where $|N^+|$ is defined by

$$\int \psi(t,q)^*\psi(t,q)dq = 1 \tag{42}$$

Taking $N,\ N^+$ both as real and positive, we have

$$N^2 \int\int e^{-q^2}dq = N^{+2}\int e^{-\frac{q^2}{2}}dq \tag{43}$$

which gives $N^2 = N^{+2}$.

One can also try to solve the Schroedinger equation directly for t¿0, given the intial condition at $t = 0^+$, but I have no idea how to do that directly.

We can use the trick to get the Schoedinger equation solution for $t = 0^+$ from problem 3, where

$$i\frac{d}{dt}\ln(\psi(t=0,q)) = \frac{1}{2}\epsilon\delta(t)q^2 \tag{44}$$

$$\ln(\psi(t=0^+,q)) - \ln(\psi(t=0^-,q) = -i\epsilon q^2 \tag{45}$$

or

$$\psi(0^+,q) = \psi(0^-,q)e^{-i\epsilon\frac{q^2}{2}} = Ne^{-\frac{q^2}{2}(1+i\epsilon)} \tag{46}$$

as the intial condition.

To first order in ϵ,this is

$$\psi(0+,q) = Ne^{-\frac{q^2}{2}}(1+i\epsilon q^2/2) \tag{47}$$

$$= Ne^{-q^2/2}[1 - i\epsilon/4 + i\epsilon(q^2 - 1/2)/2)] \tag{48}$$

The second term in parenteses is proportional to the second Hermite polynomial and the first is proportional to the 0th Hermite polynomial, so this is a sum of the lowest and the third energy eigenstates. The lowest energy eigenstate has a time dependence of $e^{-it/2}$ while the third has time dependence of $e^{-i(5/2)t}$. So to first order in ϵ the solution must be

$$\psi(t,q) = Ne^{-q^2/2}[(1-i\epsilon/4)e^{-it/2} + i\epsilon(q^2-1/2)/2)e^{-i5t/2}]] \tag{49}$$

Which is easier?
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The answer is up to you. If one simply wants to calculate the energy expectation value this one is easier. If one wants to calculate the full time dependence, the Heisenberg represenation is much easier, unless the state happens to be an energy eigenstate.

Note the state of the system after the t=1 is called a squeezed state, which has become a very powerful took in quantum optics in increasing the sensitivity of certain detectors beyond what one might naively call the quantum limit. We will look at this later in the course.

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2)Consider the Hamiltonian

$$H = \frac{1}{2}(P^2 - Q^2) \tag{50}$$

Find the solution to the Heisenberg equations of motion for this system, assuming that at $t = 0$ the operators are $P_0,\ Q_0$.

Consider the state of the system to be

$$|\psi\rangle = \int e^{-q^2/2}\,|q\rangle\, adq \tag{51}$$

In terms of the operators P_0, Q_0, what equation does this initial state satisfy? Write this equation in terms of $P(t)$ and $Q(t)$. In terms of the eigenvalues of $Q(t)$ what is the state of the system? (This is the Schroedinger equation solution.)

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The equations of motion are

$$\partial_t Q = -i[Q, H] = -i[Q, \frac{1}{2}P^2] = P \tag{52}$$

$$\partial_t P = -i[P, H] = -i[P, -\frac{1}{2}Q^2] = Q \tag{53}$$

The solution to these equation, given the conditions at $t = 0$ are

$$Q(t) = Q_0 cosh(t) + P_0 sinh(t) \tag{54}$$

$$P(t) = P_0 cosh(t) + Q_0 sinh(t) \tag{55}$$

the state is $|\psi\rangle = \frac{1}{(2\pi)^{1/4}} e^{-q^2/2} |\psi\rangle$ or in the position basis

$$\psi(q) = \frac{1}{(2\pi)^{1/4}} e^{-q^2/2} \tag{56}$$

which obeys

$$\partial_q \psi + q\psi = 0 \tag{57}$$

$$(iP_0 + Q_0)\psi = 0 \tag{58}$$

But

$$Q_0 = cosh(t)Q(t) - sinh(t)P(t) \tag{59}$$

$$P_0 = cosh(t)P(t) - sinh(t)Q(t) \tag{60}$$

so the equation for the state in terms of the dynamic variables at time t is

$$(i(cosh(t)P(t) - sinh(t)Q(t)) + (cosh(t)Q(t) - sinh(t)P(t)))\,\psi = 0 \tag{61}$$

or

$$((i\cosh(t) - sinh(t))(-i)\partial_{q_t} + (\cosh(t) - i\sinh(t))q_t)\,\psi(t, q_t) = 0 \tag{62}$$

Which has solution

$$\psi(t, q_t) = Nexp(iq_t^2/2\frac{i\cosh(t) - \sinh(t)}{\cosh(t) - i\sinh(t)} \tag{63}$$

$$= exp(-q_t^2/2\frac{cosh(t) + isinh(t)}{cosh(t) - i\sinh(t)} \tag{64}$$

$$= exp(-q_t^2 e^{i\phi(t)}) \tag{65}$$

where $tan(2\phi(t)) = tanh(t)$ This is the Schroedinger wave function in terms of the $Q(t)$ representation.

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3)Consider the Hamiltonian with dynamic variables $X,\ P$

$$H = \delta(t)X \tag{66}$$

What are the Hamiltonian representation solutions for this and what is the Schroedinger equation?

Assume that just before t=0, the wave function is $\alpha e^{-\beta x^2/2}$ where $\alpha,\ \beta$ are complex constants. In the Schroedinger representation what will the state be just after t=0? (Can ψ be continous? If not what does a delta funtion times a non-continous function mean?

What is the solution for the Schroedinger equation just after t=0 using the Heisenberg solution to find it? What does this suggest about the solution to the above problem?

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The Schroedinger equation is

$$i\partial_t\psi(t,x) = \delta(t)x\psi(t,x) \tag{67}$$

and the Heisenberg equation is

$$P(t) = -\delta(t) \tag{68}$$

which has solution

$$P(t) = P(0-) - \theta(t) \tag{69}$$

where θ is the Heviside step function ($\theta(t) = 0$ for $t < 0$ and $\theta(t) = 1$ for $t > 0$)

The Schroedinger equation gives

$$ln(\psi(0+,x)) = ln(\psi(0-,x)) - i\theta(t)x \tag{70}$$

$$\psi(0+,x) = e^{-ix}\psi(0-,x) \tag{71}$$

Note that ψ is not continuous at t=0, making the Schroedinger equation somewhat ambiguous.

However, we can look at this from the Heisenberg picture point of view. Lets assume that the state just before $t = 0$ is gaussian so $\psi(0-,x) = \alpha e^{-\beta x^2/2}$ which obeys $(iP(0-)+\beta X(1-))\psi(1-,x) = 0$ But with the Hamiltonian $H = \delta(t-1)X$, we have

$$P(1+) = P(0-) - 1 \tag{72}$$

$$X(0+) = X(0-) \tag{73}$$

or

$$P(0-) = P(0+) + 1 \tag{74}$$

$$X(1-) = X(0+) \tag{75}$$

which give the equation for the Schroedinger wavefunction at 1+ as

$$(i(P(0+) + 1) + \beta X(0+)\psi(0+, x^+) = 0 \tag{76}$$
$$\partial_{x^+}\psi(0+, x^+) + (i + \beta x^+)\psi(1+) = 0 \tag{77}$$

which has as solution

$$\psi(0+, x^+) = N^+ e^{-(ix+\beta(x+)^2} \tag{78}$$

Where N^+ is a normmalisation factor. But$|\psi|^2$ is the same function of x^- and x^+ since the added term is a pure x-dependent phase which drops out of the probability density. Thus we can take the normalisation to be α (ie continuous, as a function of t). Ie, the logarithmic derivative to find how ψ changes across the temporal delta function is the right thing to do.
