QUANTUM NOISE IN THE INTERFEROMETER DETECTOR

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In this paper I will examine the quantum noise sources in a laser interferometer detection system for gravitational radiation. The quantum noise sources will be of two basic forms - that due to the quantum nature of the light itself and that due to the damping in the mirror masses used as reflectors in the interferometers. We will find that the quantum nature of the light is the dominant source of noise and contributes via two mechanisms - directly as what has been called the photon counting noise and indirectly via the fluctuating force the light exerts on the mirrors. It will be shown that by setting up the initial state of the field entering the input port of the interferometer not being used by the laser in a generalised squeezed state, the effect of both of these noise sources can be made as small as desired. (The possibility for reducing the direct noise by a similar technique was shown by Caves[1] for a simple single mode interferometer model). The noise introduced by the damping of the motion of the mirror masses will contribute significantly only if one does squeeze the state of the light beam and if the laser power is sufficiently large.

The model for the interferometric detector is given in Figure 1. The light in the interferometer arms is modelled by massless scalar fields Φ_1 and Φ_2 where Φ_1 depends only on t and z while Φ_2 depends only on t and x. Regarding both light beams as polarised in the y direction, these scalar fields may be taken to be the y component of the vector potential over $\sqrt{2\pi}$ in a gauge where the scalar potential is zero. The gravity wave will be assumed to be travelling in the y direction with polarisation axes parallel to the x and z axes. (This is the gravity wave for which

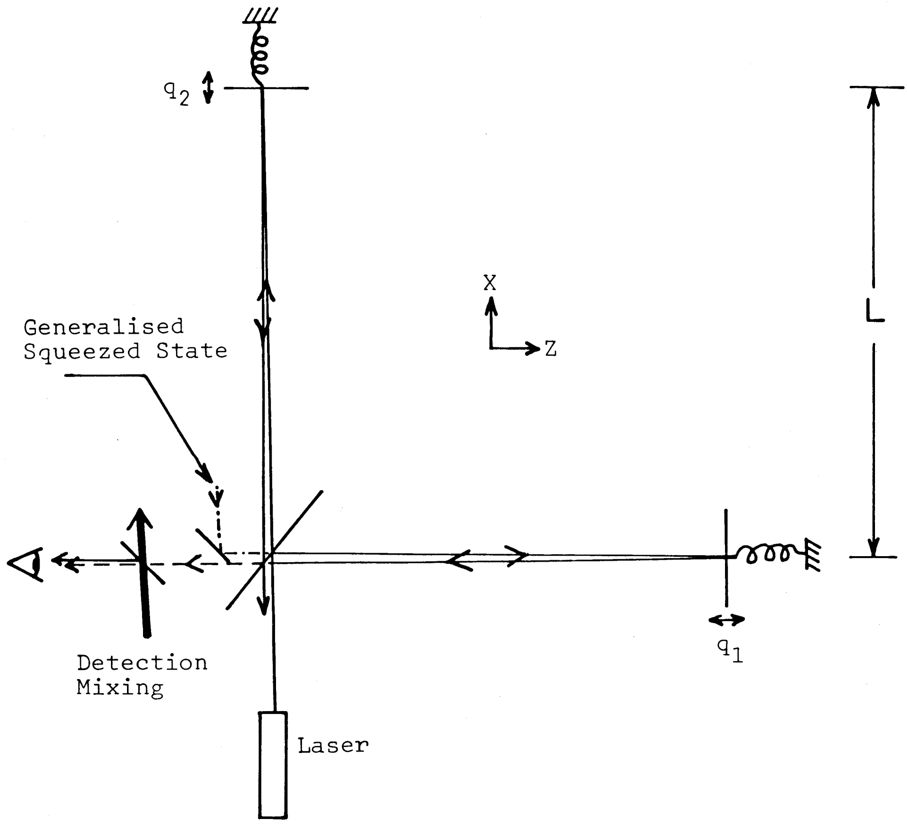
q2
X
Z
L
Generalised
Squeezed State
Detection
Mixing
q1
Laser

Figure 1

the interferometer has highest sensitivity).

The fields $\Phi(t,\xi)$, where ξ is either x or z, obey

$$\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial\xi^2} = 0 \tag{1}$$

when no gravity wave is present, with boundary conditions on the mirror of $\Phi = 0$. The number density operator if Φ are quantum fields, is given by

$$N = \frac{i}{2}\left(\Phi^{(+)}\frac{\partial\Phi^{(-)}}{\partial t} - \frac{\partial\Phi^{(+)}}{\partial t}\Phi^{(-)}\right) \tag{2}$$

where $\Phi^{(+)}$ is the positive frequency component of Φ. Finally the pressure exerted by the Φ field on any surface is given by

$$P = \frac{1}{4}\left[\left(\frac{\partial\Phi}{\partial t}\right) + \left(\frac{\partial\Phi}{\partial\xi}\right)^2\right] \tag{3}$$

(Compare this with the electromagnetic stress in the direction of motion $(E^2 + B^2)/8\pi$)

In the presence of the gravity wave[6] the equation of motion for the light changes. Let us examine the effect in the z arm. I will choose my coordinates in such a way that the metric for the gravity wave is in the transverse traceless gauge in which the end mirrors see no "forces" due to the gravity wave if they are at rest. In this coordinate system, the metric associated with the gravity wave is given by[2]

$$ds^2 = dt^2 - (1+h(t-y))\,dx^2 - dy^2 - (1-h(t-y))\,dz^2 \tag{4}$$

where h is the dimensionless strength of the gravity wave. (Note that I assume throughout that $c = \hbar = G = 1$). The equation of motion for the beam Φ_1 along the z arm becomes

$$\frac{\partial^2}{\partial t^2}\Phi_1 - (1+h(t))\frac{\partial^2}{\partial z^2}\Phi_1 = 0\ . \tag{5}$$

Assuming a solution of the form

$$\Phi_1 = \Phi_{10}\, e^{i\,\theta(t,z)}$$

where θ is a rapidly varying phase, the eikonal approximation[3] gives the equations

$$k_\mu = \theta_{,\mu} \tag{6a}$$

$$g^{\mu\nu} k_\mu k_\nu = 0 \tag{6b}$$

(The other eikonal equation $k^\mu k_{\nu;\mu} = 0$ is automatically satisfied if the above two equations are satisfied). Right and left travelling solutions are given to lowest order in h by

$$\theta = \theta_{(\pm)}(t\pm z) + \frac{1}{2} \int h(t-y)\, \theta'_{(\pm)}(t-z) - \frac{1}{2} \int h(t\pm z-y)\, \theta'_{(\pm)}(t\pm z) \tag{7}$$

where the prime denotes the derivative, $\theta_{(\pm)}$ are arbitrary functions and the function $\int h$ is defined by

$$\int h(\omega) = \int_0^\omega h(\xi)\, d\xi \tag{8}$$

The solution for Φ along the y = 0 plane is now given by

$$\Phi = \Phi_0 \,\text{expi}\, (\theta_-(t-z) + \frac{1}{2}(\int h(t) - \int h(t-z))\, \theta'_-(t-z)) + \tilde{\Phi}_0 \,\text{expi}\, (\theta_+(t+z) + \frac{1}{2}(\int h(t) - \int h(t+z))\, \theta'_{(+)}(t+z)) \tag{9}$$

The boundary condition at the mirror located at $z = L+q_1$ (where L is the equilibrium position of the mirror) is that Φ is zero there. Therefore we have

$$\tilde{\Phi}_0 = - \Phi_0 , \tag{10}$$

$$\theta_+(t+L+q(+)) + \frac{1}{2}(\int h(t) - \int h(t+L+q))\, \theta'_+(t+L+q) = \theta_-(t-L-q(t)) + \frac{1}{2}(\int h(t) - \int h(t-L-q))\, \theta'_-(t-L-q)$$

If we assume q(t) to be very small such that terms of order q^2 or qh can be neglected, and if we assume $\dot{q} << 1$ (where the dot denotes time derivative), the solution is

$$\theta_+(t) = \theta_-(t-2L)\, \frac{1}{2}(\int h(t,0) - \int h(t-2L,0))\, \theta'_-(t-2L) - 2\, q(t-L)\, \theta'_-(t-2L) \tag{11}$$

where I have also dropped terms of order h^2.

The net phase shift when the light returns to z=0, over what it would have been had h = q = 0, is given by

$$\Delta\, \theta(t) = (-2q(t-L) + \frac{1}{2} \int h(t) - \int h(t-2L))\, \theta'_-(t-2L) \tag{12}$$

The net phase shift is thus given by a combination of the motion of the mirror (noise) and the effect of the gravity wave on the light (signal).

In the x arm, which I will assume to have the same length L, the phase shift due to the gravity wave is minus that in the z arm. Assuming that $\Delta\theta$ in both arms is small, recombining the two beams at x = z = 0 by means of a beamsplitter and looking at the difference port we have a signal there of

$$\begin{aligned}\Phi_T = \Phi_D\, e^{i\theta(t-2L)} + \\ \frac{1}{\sqrt{2}}\big(-2q_1(t-L)\ \theta'(t-2L)\ \Phi_{01} \\ +2q_2(t-L)\ \theta'(t-2L)\ \Phi_{02}\big) \\ + \frac{1}{2}\big(\int h(t) - \int h(t-2L)\big)\ \Phi_s \ \}\end{aligned} \tag{13}$$

where

$$\Phi_D = (\Phi_{01} - \Phi_{02})/\sqrt{2}$$

$$\Phi_s = (\Phi_{01} + \Phi_{02})/\sqrt{2}$$

(The beamsplitter is assumed to transmit the sum and difference signals over $\sqrt{2}$. The $\sqrt{2}$ is necessary to conserve the total particle number).

The above has been a classical analysis of one mode of the field which begins with phase $\theta(t)$ at z=0. I will now quantise the system. Φ_1, Φ_2 now become quantum Heisenberg operator, as does q. h is assumed to be a classical field. The above solution will be assumed to also be sufficiently good approximation to the solution for the quantum system. (Statements like "q or $\dot{q}$ is small" now mean that in all states of the system of interest the expectation value of q or $\dot{q}$ and expectation values of relevant powers of these operators are sufficiently small, or,the probability that q is larger than a fraction of wavelength of any light frequency of interest or that $\dot{q}$ is not much less than the velocity of light is small).

We will characterise the various modes of the light by their frequency ω on entering the interferometer at z=0. In this case $\theta(t)$ is ωt and θ' is ω. Furthermore, the laser is assumed to put out light at a fixed frequency ω_0 with amplitude $\sqrt{2}$ A and all other frequencies are assumed to have negligible intensity. The laser is assumed to shine into the "sum" input port, so that

$$\langle\Phi_1(t)\rangle = \langle\Phi_2(t)\rangle = (Ae^{-i\omega_0 t} + c.c.)$$

(i.e. both arms of the interferometer have light of the same intensity and phase entering them).

We can now write the input field Φ_{Ii} (i=1,2) in either arm at x = z = 0 as

$$\Phi_{Ii}^{(+)} = \{(A\, e^{i\omega_o t} + \int_{\omega>o} \hat{\Phi}_i(\omega)\, e^{i\omega t}) + \text{herm. conj.}\} \qquad (14)$$

where I will again assume the quantum operators $\hat{\Phi}$ are "small" with respect to A.

At the output port, the operator output signal will now be given by

$$\begin{aligned}\Phi_T &= (\int \hat{\Phi}_D(\omega)\, e^{i\omega(t-2L)}\, d\omega + \text{Herm. conj.}) \\ &+ 2(q_2(t-L) - q_1(t-L))(A\, e^{i\omega_o(t-2L)} - c.c.) \qquad (15)\\ &+ \frac{1}{2}(\int h(t) - \int h(t-2L))\ (A\, e^{i\omega_o(t-2L)} - c.c.)\end{aligned}$$

where products of $\hat{\Phi}$ with q_1 or q_2 have been dropped.

This expression has two noise components and one signal. The signal is the term proportional to the gravity wave. The noise is of two forms. One is the noise due to the quantum nature of the light itself (terms prop. to $\hat{\Phi}_D$) and the other is the noise due to the motions of the mirrors. These two noise sources are not strictly independent however. Part of the mirror motion is due to the fluctuations in the radiation pressure on the mirrors, which ultimately is due to the quantum nature of the light. Instead of analyzing the noise at this stage, it will be more illuminating to study the motion of the mirrors first. Before we do that however, let us decide on what we are going to do with the output signal Φ_T. We must decide what property of this quantum operator we wish to measure. Since virtually the only light detector is a photomultiplier, I will assume that we will measure the number flux operator of the outgoing field. Furthermore, since I want to measure h, not h^2, I will mix the output field with a "classical" field of amplitude ($B\, e^{i\omega_o(t-2L)}$+ c.c.) and then detect the number flux out of the interferometer. Although there is no strictly classical field, we could mix Φ_T with a quantum field χ with a very small mixing angle, α. The measured field will be

$$\begin{aligned}\Phi_M &= \cos\alpha\ \Phi_T + \sin\alpha\ \chi \qquad (16)\\ &= \cos\alpha\ \Phi_T + \sin\alpha\left(\frac{B}{\sin\alpha}\, e^{-i\omega(t-2L)} + c.c. + \hat{\chi}\right)\end{aligned}$$

where $\hat{\chi}$ is the quantum part of the χ field. If α is sufficiently small we can make the effect of $\hat{\chi}$ on any measured property as small as we want.

The number flux operator at time t for the field Φ_M is given by

$$N(t) = \frac{i}{2} [\Phi_M^{(+)}(t)\ \dot{\Phi}_M^{(-)}(t) - \dot{\Phi}_M^{(+)}(t)\ \Phi_M^{(-)}(t)]$$
$$\approx \omega_o\ |B|^2 + \frac{i}{2} (B^* e^{+i\omega_o(t-2L)}(-i\omega_o\ \Phi_T^{(-)} + \dot{\Phi}_T^{(-)}) \qquad (17)$$
$$+ B\ e^{-i\omega_o(t-2L)}(-i\omega_o\ \Phi_T^{(+)} - \dot{\Phi}_T^{(+)}))$$

I have assumed that B is sufficiently large that terms of order $\Phi_T{}^2$ can be neglected. Substituting the expression for Φ_T in terms of the Fourier components of $\hat{\Phi}$ and q_i we finally have

$$N(t) = \{\omega_o\ |B|^2$$
$$+ \int \{B^*(\omega_o + \tfrac{\mu}{2})\ \hat{\Phi}_D(\omega_o+\mu) + B(\omega_o - \tfrac{\mu}{2})\ \hat{\Phi}_D^{\dagger}(\omega_o-\mu)\ e^{-i\mu(t-2L)}\}\, d\mu$$
$$(18)$$
$$+ 2i \int \omega_o\{B^* A(\omega_o + \tfrac{\mu}{2})\ q_D(\mu) - BA^*(\omega_o - \tfrac{\mu}{2})\ q_D(\mu)\}\ e^{-i\mu(t+L)} d\mu$$
$$-\ i \int \omega_o\{B^*\ A(\omega_o + \tfrac{\mu}{2})\ Ih(\mu) - BA^*(\omega_o - \tfrac{\mu}{2})\ Ih(\mu)\}\ e^{-\mu t} d\mu$$

where

$$Ih(\mu) = (\frac{i}{2\mu}\ (1 - e^{2i\mu L})\ h(\mu)) \qquad (19a)$$
$$q_D(t) = q_1(t) - q_2(t) = \int q_D(\mu)\ e^{i\mu(t)}\ d\mu \qquad (19b)$$
$$\hat{\Phi}_D(t) = \frac{1}{\sqrt{2}}\ (\hat{\Phi}_1(t) - \hat{\Phi}_2(t)) = \frac{1}{\sqrt{2}} \int \hat{\Phi}(w)\ e^{-i\omega t}\ d\omega \qquad (19c)$$

To evalulate the q_i we need a model for the motion of the masses in the z or x direction. Let us again concentrate on the z arm mass. I will assume that the mirror is suspended as simple harmonic oscillators with respect to its motion in the z direction. Furthermore, it is damped by means of a coupling to some damping field ψ. The interaction of the light beam with the mirror masses is by means of the radiation pressure of the light on the mirror. The resultant equation of motion of the mirror in this simple model is given by

$$M\ddot{q} + \left(\frac{\varepsilon^2}{2} + 2\,\omega_o^2\,|A|^2\right)\dot{q} + M\Omega^2 q = -\varepsilon\,\psi_o(t) + 2\,\omega_o^2\,|A|^2$$

$$- 2\,\omega_o \int ((\omega_o-\mu)\,A\,\hat{\Phi}^\dagger(\omega_o-\mu) + A^*(\omega_o+\mu)\,\hat{\Phi}(\omega_o+\mu))e^{-i\mu(t-L)}d\mu \tag{20}$$

(In deriving this expression I have used the model coupling to the ψ field of Unruh (1980)[4].) In the expression for the pressure on the mirror by the light (eq. 3), I have neglected all terms which are smaller than first order in A and higher order than first in q.

The term proportional to $|A|^2$ is a constant force on the mirror and could be eliminated by redefining the equilibrium position for the mirror or by imposing another equal but opposite force on the mirror. The two terms in ψ_o and $\hat{\Phi}$ represent quantum fluctuating forces on the mirrors due to the dampling field and the light beams respectively. Note that the damping coefficient of the oscillator depends both on the damping field and the light beam. This damping by the light can be easily understood. If the oscillator has a velocity $\dot{q}$, the light reflected from the mirror will suffer a red shift of $\omega_o(2\,\dot{q}/c)$. Each outgoing photon will therefore have $\hbar\,\omega_o(2\,\dot{q}/c)$ less momentum than it would have if $\dot{q}$ were zero and will therefore have transfered that much less momentum to the mirror. If there are n photons per second, this will correspond to a force which is $n\,\hbar\,\omega_o(2\,\dot{q}/c)$ less than if the mirror were at rest. Since $|A|^2\,\omega_o$ equals n we obtain the same damping coefficient as in the above equation.

We can solve this equation for the Fourier component, $q_i(\mu)$, of the displacement. In particular we will be interested in those frequencies which are much larger than the fundamental frequency of the mirrors, Ω. (I will assume both mirrors have the same frequency). The solution is

$$q(\mu) = \{i\mu\varepsilon\,\psi(\mu) - 2\,\omega_o(A(\omega_o-\mu)\hat{\Phi}^\dagger(\omega_o-\mu) + A^*(\omega_o+\mu)\hat{\Phi}(\omega_o+\mu))$$
$$\times e^{+i\mu L}\}/\Sigma(\mu)$$

$$\Sigma(\mu) = (-M\mu^2 - i\mu\left(\frac{\varepsilon^2}{2} + 2\,\omega_o^2|A|^2\right) + \Omega^2) \tag{21}$$
$$\approx -M\mu^2$$

Substituting this expression for the q of each mirror into the equation for N(t) we finally obtain (after using $\mu \ll \omega_o$)

$$
\begin{aligned}
e^{-2i\mu L}N(\mu) \approx\ & |B|^2\, \omega_o \delta(\mu) \\
& - i\ \frac{(B^*A - BA^*)}{\sqrt{2}}\ \omega_o{}^2\, \frac{2\varepsilon}{M\mu}\, \psi_D(\mu)\ e^{-i\mu L} \\
& - \omega_o\ \{\ \hat{\Phi}_D(\omega_o+\mu)\ [\ B^* + \frac{4i\omega_o{}^3}{M\mu^2}\ (B^*|A|^2 - BA^{*2})\ ] \\
& \quad + \hat{\Phi}_D{}^{\dagger}\ (\omega_o - \mu)\ [B + \frac{4i\omega_o{}^3}{M\mu^2}\ (B|A|^2 - B^*\ A^2)]\} \\
& - \frac{i(B^*A - BA^*)\ \omega_o{}^2}{2\sqrt{2}}\quad Ih(\mu)e^{-i2\mu L}
\end{aligned}
\tag{22}
$$

where $\psi_D = \frac{1}{\sqrt{2}}\ (\psi_{10} - \psi_{20})$

To maximize the sensitivity we must obviously choose B^*A to be purely imaginary i.e.

$$B^*A = i\ |B|\ |A| \tag{23}$$

This finally results in the following expression for the μ^{th} Fourier component of the number flux

$$
\begin{aligned}
e^{-2i\mu L}\ N(\mu) =\ & |B|^2\ \omega_o\delta(\mu)\ +\ |B|\,|A|\ \frac{2\omega_o{}^2\varepsilon}{M\mu}\ \psi_D(\mu) \\
& + \omega_o\ \{\ \hat{\Phi}_D(\omega_o+\mu)\ B^*[\ 1 + \frac{8i\omega_o{}^3}{M\mu^2}\ |A|^2\ ] \\
& \quad + \hat{\Phi}_D{}^{\dagger}(\omega_o - \mu)\ B\ [1 - \frac{8i\omega_o{}^3}{M\mu^2}\ |A|^2]\} \\
& + \sqrt{2}\ |B|\,|A|\ \omega_o{}^2\ \ Ih(\mu)\ e^{-2i\mu L}
\end{aligned}
\tag{24}
$$

It is clear that to make the quantum noise of the light small we must choose the initial state of the Φ_D field such as to make the fluctuations of the operators

$$\hat{\Phi}(\omega_o + \mu)\ e^{i\lambda(\mu)}\ +\ \hat{\Phi}^{\dagger}(\omega_o - \mu)\ e^{-i\lambda(\mu)}$$

small for all frequencies μ of interest. Here I have defined $\lambda(\mu)$ by

$$\lambda(\mu) = \arg\ [\ B^*\ (1 + 8i\ \frac{\omega_o{}^3\ |A|^2}{M\mu^2})\ ] \tag{25}$$

This is possible if we place the Φ_D field into the generalized squeezed state $|GS>$, defined by[5]

$$\{\cosh r(\mu)\hat{\Phi}(\omega_o+\mu)e^{i\lambda(\mu)} + \sinh r(\mu)\hat{\Phi}^{\dagger}(\omega_o-\mu)e^{-i\lambda(\mu)}\}|GS> = 0 \quad (26)$$

Note that this is a true squeezed state[1] only if $\mu = 0$. At other frequencies this state contains correlations between the frequencies $\omega_o+\mu$ and $\omega_o-\mu$. If the Φ_D field is placed into one of these generalized squeezed states, and the ψ_o field is assumed to be in its vacuum state, we can calculate the fluctuations in the Fourier components of the number operator. We have

$$< N(\mu) > = |B|^2 \omega_o \delta(\mu) + \frac{|B||A|}{\sqrt{2}} Ih(\mu) e^{-2i\mu L} \quad (27)$$

and

$$< N(\mu) N(\mu')> - < N(\mu) > < N(\mu') > =$$

$$8|B|^2|A|^2 w_o^4 \frac{\varepsilon^2}{(M\mu)(M\mu')} < \psi_D(\mu) \psi_D(\mu') >$$

$$+ \frac{\omega_o^3}{2} |B|^2 \left| 1 + \frac{8i\omega_o^3 |A|^2}{M\mu^2} \right| \left| 1 + \frac{8i\omega_o^3 |A|^2}{M\mu'^2} \right|$$

$$\times < (\hat{\Phi}(\omega_o+\mu)e^{i\lambda(\mu)} + \hat{\Phi}^{\dagger}(\ _o-\mu)e^{-i\lambda(\mu)})(\hat{\Phi}(\omega_o+\mu')e^{i\lambda(\mu')} + \hat{\Phi}^{\dagger}(\omega_o-\mu')e^{-i\lambda(\mu')}) > \quad (28)$$

$$= \delta(\mu+\mu')\{ \frac{8\omega_o^4 \varepsilon^2 |A|^2|B|^2}{(M\mu)^2} \frac{2}{2\pi\mu} \theta(\mu) + \frac{\omega_o^2}{2} |B|^2 (1 + (\frac{8\omega_o^3|A|^2}{M\mu^2})^2 \frac{e^{-2r(\mu)}}{2\pi\omega_o}\}$$

where I have used the relations

$$\psi_D(-\mu) = \psi_D^{\dagger}(\mu) \; ; \; [\psi_D(\mu), \psi_D^{\dagger}(\mu)] = \frac{\delta(\mu-\mu')}{2\pi\mu}$$

$$[\Phi_i(\omega_o+\mu), \Phi_i^{\dagger}(\omega_o+\mu')] = \frac{\delta(\mu-\mu')}{2\pi(\omega_o+\mu)} \simeq \frac{\delta(\mu-\mu')}{2\pi\omega_o} \quad (29)$$

Note that by choosing $r(\mu)$ sufficiently large the term due to the quantum nature of the light can be made as small as we want.

To obtain some feeling for what this expression means physically, let us substitute some numbers for a realistic interferometer into this expression. The total power P going into the interferometer is given by (remembering that I have used $\hbar = c = 1$ to derive the above expression).

$$\mathcal{P} = 2\omega_o^2 \, |A|^2 \tag{30}$$

We have (after reinstating c and h into the expressions) and defining $\gamma_d = \varepsilon^2/4M$,

$$\begin{aligned} &\langle N(\mu)\, N(\mu')\rangle - \langle N(\mu)\rangle\langle N(\mu')\rangle \\ &= \delta(\mu+\mu')\, |B|^2\, \omega_o \left\{ \frac{32\,\gamma_d}{\pi\mu} \frac{\omega_o}{\mu} \frac{\mathcal{P}}{Mc^2\mu}\, \theta(\mu) \right. \\ &\qquad \left. + \frac{e^{-2r(\mu)}}{2\,\pi} \left(1 + \left(\frac{4\omega_o\, \mathcal{P}}{\mu\, Mc^2\mu}\right)^2 \right) \right\} \end{aligned} \tag{31}$$

There are two power levels of interest: The first is when the quantum noise introduced because the fluctuating radiation pressure driving the mirrors equals that due to the direct noise

$$\mathcal{P}_1 = (Mc^2\,\mu)\, \frac{\mu}{4\,\omega_o} \tag{32}$$

and the other when the noise due to the damping of the oscillator equals the direct noise

$$\mathcal{P}_2 = \frac{\mu}{16\gamma_d} \left(\frac{\mu}{4\,\omega_o} (Mc^2\mu) \right) = \frac{\mu}{16\gamma_d}\, \mathcal{P}_1 \tag{33}$$

Since by assumption $\mu \gg \gamma_d$, $\mathcal{P}_2$ will be larger than $\mathcal{P}_1$. This implies that the quantum nature of the light is responsible for the dominant noise at all power levels if the initial state is unsqueezed. For $M = 1$ kg, $\mu \sim 10^4$ hz, $\omega_o \sim 10^{15}$ hz, we have $\mathcal{P}_1 \approx 2.5 \times 10^9$ watts. Therefore, at all forseeable power levels for real interferometers, the dominant noise source will be the direct quantum light noise (which has been called the photon counting noise by Caves). Only if the light state is strongly squeezed will the damping fluctuations of the mirror become important.

If the damping bath due to the ψ fields is thermal with a non zero temperature T, the first term is multiplied by a factor of

$$\beta_T = (1 + 2/(e^{\hbar\mu/kT} - 1)) \tag{34}$$

For sufficiently high temperatures this becomes $2kT/\hbar\mu$ which could result in a significant noise from this damping field. For frequencies $\mu = 10^4$ hz and T = 300 K, this factor is 8×10^9. For $\gamma_d = 10^{-4}$/sec, the power at which this noise source becomes significant is

$$\mathcal{P}_{2T} = \frac{10^4}{16 \times 10^{-4}} \; 2.5 \times 10^9 / 8 \times 10^9$$
$$= 2 \times 10^6 \text{ watts}$$

again a rather unrealistic figure. Even if the mirrors were critically damped and had a period of 1 sec, the power level for this source of noise to be most important would still be about 100 watts. (The effective temperature of seismic and suspension noise on the mirrors could of course be much greater than room temperature, reducing this noise will be one of the chief experimental difficulties).

Finally we will not be measuring $N(\mu)$ directly. Rather we will be measuring some frequency component μ_o of $N(t)$ over some time period τ. Furthermore, we will be measuring either the sine or cosine phase. We have

$$N_c\,(\mu_o, T) \approx \int_T^{T+\tau} N(t) \cos \mu_o t \, dt$$

and similarly for the sine component, N_s. Evaluating $(\Delta N_c)^2$ and $(\Delta N_s)^2$ we find they are both of order

$$(\Delta N)^2 \approx \frac{\tau |B|^2 \omega_o}{4\pi} \left\{ \frac{16\gamma_d \omega_o \mathcal{P}}{\mu_o^2 (Mc^2 \mu_o)} \beta_T + \frac{e^{-2r(\mu_o)}}{2} \left[1 + \left(\frac{4\omega_o \mathcal{P}}{\mu_o (Mc^2 \mu_o)} \right)^2 \right] \right\} \qquad (35)$$

while the signal is of order

$$\langle N(\mu_o, T) \rangle \sim \omega_o^2 |B| |A| (L/c) h(\mu_o, T) \; \tau .$$

The signal to noise ratio is therefore of order

$$\frac{\sqrt{\omega_o \tau} \;\; \sqrt{\mathcal{P}/h\omega_o^2} \;\; (\omega_o L/c) \;\; h(\mu_o, T)}{\sqrt{\frac{e^{-2r}}{2} \left[1 + \left(\frac{4\omega_o \mathcal{P}}{\mu_o (Mc^2 \mu_o)} \right)^2 \right] + \frac{16\gamma_d \omega_o \mathcal{P}}{\mu_o (Mc^2 \mu_o)} \beta_T}} \qquad (36)$$

where I have dropped factors of order unity.

One important point to mention is that the above expression for the noise is valid only as long as the terms proportional to $\hat{\Phi}_T{}^2$ can be neglected. As r gets larger (higher squeezing) these terms become more important. Crude estimates suggest that once $e^r > |B|^{2/3}$, these terms will dominate the terms we have retained and the noise will again increase with increasing r. Neglecting the noise due to the damping field, this suggests that the maximum signal to noise increases only as $B^{1/3}$ or $(P_B)^{1/6}$ with the optimal value of the squeezing parameter r.

In the above I have given a prescription for a generalised squeezed state which obeys an equation like eq. 26. Is such a generalised squeezed state possible? The answer is yes, and it should be realised by a three wave mixer. Consider a non linear medium which has an interaction with light such that the energy has a term which goes as $\varepsilon\Phi^3$. Supply a pump wave with frequency $2\omega_o$ and amplitude $\hat{A}$. The linearised equations of motion for the quantum field $\hat{\Phi} = \Phi - (A\, e^{-i2\omega 0(t-x)} + c.c.)$ are of the form

$$\frac{\partial^2\hat{\Phi}}{\partial t^2} - \frac{\partial^2\hat{\Phi}}{\partial x^2} = \varepsilon\, \hat{\Phi}(Ae^{-i\omega_o(t-x)} + c.c.). \tag{37}$$

Writing $\hat{\Phi} = \int\hat{\Phi}(\mu,t)\, e^{-(\omega_o+\mu)(t-x)} d\mu$ where $\hat{\Phi}(\mu,t)$ is slowly varying in time, we have

$$\begin{aligned} -2i(\omega_o+\mu)\frac{\partial\hat{\Phi}}{\partial t}(\mu,t) &\approx \varepsilon A\hat{\Phi}(-2\omega_o+\mu,t) \\ &= \varepsilon A\hat{\Phi}^\dagger(-\mu,t) \end{aligned} \tag{38}$$

The solution is

$$\hat{\Phi}(\mu,t) = \cosh\sigma t\, \hat{\Phi}(\mu,0) + \frac{iA}{|A|} \sinh\sigma t\, \hat{\Phi}^\dagger(-\mu,0) \tag{39}$$

with

$$\sigma = \left(\frac{\varepsilon|A|}{2\omega_0}\right).$$

If the waves travel in the medium for time T and the initial state of the $\hat{\Phi}$ field is the vacuum so that

$$\hat{\Phi}(\mu,0)\,|0\rangle = 0$$

then the stae $|0\rangle$ will be a generalised squeezed state for the modes leaving the medium whose frequencies are near ω_o with the squeezing parameter r given by σT.

In conclusion, the dominant quantum noise source in the interferometric readout is that due to the quantum nature of the light. This noise source can be reduced an arbitrary amount by placing the state going into the non laser input port of the interferometer into a generalised squeezed state.

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5. An analysis of a "generalized squeezed state" (called a "multimode squeezed state" by Caves) is given in an appendix to C. M. Caves "Quantum Limits on Noise in Linear Amplifiers" preprint, California Institute of Technology, Pasadena, California.

6. A similar analysis has been performed by A. D. Alekseev, L. F. Vitushkin, N. I. Kolosnitsyn, V. M. Moskovkin, JETP 52 588 (1981). Original in Zh. E.T.F. 79 1141 (1980).