Physics 501
Midterm Exam
Feb 28 2022

This exam is 1hr 15 min (75 min) in length. It is closed book.

This exam has 5 questions, all of equal value.

When the exam has been marked, and handed back to you, you will have the option of redoing any of the questions on which you did not get full marks and handing the result in for remarking. As your midterm mark you will get the average of the mark you received on the midterm writing itself and the mark you get on the redo of the question, but in no case will you get less than that the mark you received on the midterm marking itself. This "re-do" will be due 1 week after the results of the marking of the original midterm are sent back to you, with no extention of that time. You may discuss this redo with others, and use any notes, text-books, etc. in doing so, except you may not simply copy someone else's solution(s).

1. Consider the sigma matrix

$$\sigma_\theta = cos(theta)\sigma_z + sin(\theta)\sigma_x \tag{1}$$

and the Hamiltonian for this system of

$$H = 0 \tag{2}$$

At 9AM σ_z was measured and found to have a value of +1. At 11AM σ_x was measured and found to have a value of +1. At 10AM σ_θ was measured .

a) What is the probability that the value of +1 was found this measurement,as a function of θ given the above conditions.

$$\mathcal{P}_{\theta,1} = |\langle x,1|\,|\theta,1\rangle|^2 |\langle\theta,1|\,|z,1\rangle|^2/N \tag{3}$$

$$\mathcal{P}_{\theta,-1} = |\langle x,1|\,|\theta,-1\rangle|^2 |\langle\theta,-1|\,|z,1\rangle|^2/N \tag{4}$$

where N is chosen so that the sum of the two probabilities is 1.

Now assuming $|z,1\rangle = |+\rangle$ and $|z,-1\rangle = |-\rangle$ then

$$|x,1\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \tag{5}$$

$$|x,-1\rangle = \frac{1}{\sqrt{2}}(|+\rangle - |-\rangle) \tag{6}$$

and

$$|\theta,1\rangle = (cos(\theta/2)\,|+\rangle + sin(\theta/2)\,|-\rangle) \tag{7}$$

$$|\theta,-1\rangle = (cos(\theta/2)\,|-\rangle - sin(\theta/2)\,|+\rangle) \tag{8}$$

Thus

$$\mathcal{P}_{\theta,1} = |\frac{1}{\sqrt{2}}(cos(\theta/2) + \sin(\theta/2))|^2 |cos(\theta/2)|^2/N \quad (9)$$

$$= \frac{1}{2}(1 + sin(\theta)cos^2(\theta/2)/N\mathcal{P}_{\theta,1} = |\frac{1}{\sqrt{2}}(cos(\theta/2) - sin(\theta/2))|^2 |sin(\theta/2)|^2/N \quad (10)$$

$$= \frac{1}{2}(1 - sin(\theta))sin^2(\theta/2) \quad (11)$$

and

$$N = \frac{1}{2}((1 + sin(\theta))(cos^2(\theta/2) + (1 - sin(\theta)(sin^2(\theta)) \quad (12)$$

$$= \frac{1}{2}(1 + sin(\theta)(cos^2(\theta/2) - sin^2(\theta/2)) \quad (13)$$

$$= \frac{1}{2}(1 + sin(\theta)cos(\theta)) = \frac{1}{2}(1 + \frac{1}{2}sin(2\theta)) \quad (14)$$

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b) If σ_θ was weakly measured at time 10AM instead of exactly measured, what was the weak expectation value of σ_θ as a function of θ.

The weak value is an operator A is given by

$$W(A) = \langle f| A |i\rangle / |f\rangle |i\rangle \quad (15)$$

where $|i\rangle$ is the intial state and $|f\rangle$ is the final state.

In this case $|i\rangle = |+\rangle$; $\quad |f\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle)$ and thus

$$W(A) = (\langle +| + \langle -|)(cos(\theta)\sigma_z + sin(\theta)\sigma_x) |+\rangle / \langle +| |+\rangle \quad (16)$$

$$= cos(\theta) + \sin(\theta) \quad (17)$$

since$\sigma_x |+\rangle = |+\rangle$ and $(\langle +| + \langle -|)\sigma_x = (\langle +| + \langle -|)$

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2. Consider the Hamiltonian

$$H = -\frac{1}{2}p^2 \quad (18)$$

Consider the complex putative solution of the classical equations

$$x = 1 + it \quad (19)$$

a)Show that this is indeed a solution of the classical equations of motion. What is p for this mode?

The equations of motion are

$$\partial_t x = -p \quad (20)$$

$$\partial_p = 0 \quad (21)$$

Thus this function clearly satisfies the equations of motion with $p = -\partial_t x = -i$.
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b)What is the norm for this mode?

$$< x, x >= i(x^* p - p^* x) = 2 \tag{22}$$

Thus the normalized mode is

$$x = \frac{1}{\sqrt{2}}(1 + it) \tag{23}$$

$$p = -\frac{1}{\sqrt{2}} i \tag{24}$$

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c)What is the annihilation operator corresponding to this solution in terms of the quantum momentum and position operators at time $t = 0$ and what is the annihilation operator in terms of the quantum momentum and position operators at time t?

The quantum equations of motion are

$$\partial_t X = -P \tag{25}$$

$$\partial_t P = 0 \tag{26}$$

which has solutions

$$X = X_0 - P_0 t \tag{27}$$

$$P = P_0 \tag{28}$$

The annihilation operators is

$$< x, X >= ix^* P - p^* X = i((1 - it)P - iX) = i((1 - it)P_0 - i(X_0 + tP - 0) = (iP_0 + X_0) \tag{29}$$

3. Give the argument from Hardy's chain, that classical Mechanics cannot mimic the results from a quantum system. Go into some detail.

The essential point is that for two systems, we have that a true statement about the value of an operator A of the first system always implies the true of the value of an operator C of the second. The truth of the C of the second always implies the truth B a value for the first. The truth of the value of B always implies the truth of D of the second. But the truth of A almost never implies the truth D of the second. Ie,

$$\mathcal{A} \rightarrow \mathcal{C} \rightarrow \mathcal{B} \rightarrow \mathcal{D} \tag{30}$$

But A almost never implies D

To amplify, consider the state

$$|\psi\rangle = \sin(\theta)\,|+\rangle\,|1\rangle + \sin(\theta)\,|-\rangle\,|1\rangle + \sqrt(1-2\sin(\theta)^2) \tag{31}$$

Then if we take A to be that the measurement on the system of the state

$$|A\rangle = |+\rangle \tag{32}$$

implies that

$$|C\rangle = |1\rangle \tag{33}$$

always. Similarly if $|C\rangle$ is measured, then

$$|B\rangle = \frac{1}{\sqrt{2}}(|+\rangle + |-\rangle) \tag{34}$$

is always obtained. Finally if $|B\rangle$ then

$$|D\rangle = \frac{1}{\sqrt{2sin(\theta)^2 + \frac{1}{2}(1-2\sin^2(\theta))}sin(\theta)\sqrt{2}\,|1\rangle + \frac{\sqrt{1-2\sin(\theta)^2}}{\sqrt{2}}\,|0\rangle} \tag{35}$$

Now if A is measure, then the state of the second system is $|1\rangle$, and The probability that you would measure D would be

$$\mathcal{P} = \frac{4sin^2(\theta)}{1+2sin^2(\theta)} \tag{36}$$

As θ goes to 0, the probability goes to 0.

Ie, if we assume that , if always a measurement gives a certain value, then classical physics would imply that it always has that value. But this is violated by this system.

It is not necessary that all of the details are present for full marks. But the logical chain should be there, and that there exists some state for which the probability of the D given A becomes very small.

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4. Show that the unitary matrix

$$U = e^{i(\alpha P - \beta X)} \tag{37}$$

is a translation operator on both X and P (Ie, $U^\dagger X U = X + \mu$ for some value of μ, and similarly for P) P and X are the usual momentum and position operators. What linear combination of the operators X and P does this operator not change at all?

There are a few ways of doing this. One is to expand U

$$U^\dagger = \sum_n (\frac{1}{n!}(-i)^n(\alpha P - \beta X)^n \tag{38}$$

So

$$XU = (XU - UX) + UX \sum_n \frac{1}{n!}(-i)^n \sum_{r=1}^{n} (\alpha P - \beta X)^r [X, \alpha P - \beta X](\alpha P - \beta X)^{n-r} \quad (39)$$

$$= \alpha i \sum_n \frac{1}{(n-1)!} i^{n-1} i((\alpha P - \beta X)^{n-1} + UX = -\alpha U + UX \quad (40)$$

so

$$U^\dagger X U = (-\alpha + X) U^\dagger U = (-\alpha + X) \quad (41)$$

Similarly

$$U^\dagger P U = -beta + P \quad (42)$$

Now, $\alpha P - \beta X$ commutes with U, so U does not alter it.

Another way is to use the Campbell Baker Hausdorf relation Since $[X, P]$ is a c-number we have

$$U = e^{i(\alpha P - \beta X)} = e^{i\alpha P} e^{-i\beta X} e^{[i\alpha X, -i\beta P]/2} = e^{i\alpha P} e^{-i\beta X} e^{i\alpha\beta}/2 \quad (43)$$

Then

$$U^\dagger X U = e^{i\beta X} e^{-i\alpha P} e^{i\alpha\beta/2} X e^{-i\alpha\beta/2} e^{i\alpha P} e^{-i\beta X} \quad (44)$$

$$= e^{i\beta X}(e^{-i\alpha P} X e^{i\alpha P}) e^{-i\beta X} \quad (45)$$

$$e^{i\beta X}(X - \alpha) e^{-i\beta X} = X - \alpha \quad (46)$$

And similarly for P

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5. Consider the state for two particles, with σ_3 eigenstates $|+\rangle, |-\rangle$ for the first particle and $|1\rangle, |0\rangle$ for the second:

$$|\psi\rangle = \frac{1}{\sqrt{2}} |+\rangle |1\rangle + \frac{1}{2}(|-\rangle |1\rangle + |-\rangle |0\rangle) \quad (47)$$

a) What is the density matrix for the second particle.o

$$\rho = |\psi\rangle \langle\psi| = \frac{1}{2}(|+\rangle |1\rangle + \frac{1}{\sqrt{2}}(|-\rangle |1\rangle + |-\rangle |0\rangle))(\langle +| \quad (48)$$

$$bra1 + \frac{1}{\sqrt{2}}(\langle -| |1\rangle + \langle -| \langle 0|) \quad (49)$$

$$(50)$$

Then the trace over the first particle, recalling that $\langle +| |+\rangle = \langle -| |-\rangle = 1; \quad \langle +| |-\rangle = 0$ give

$$TR_1\rho = \frac{1}{2}(|1\rangle\langle 1|) + \frac{1}{4}(|1\rangle + |0\rangle)(\langle 1| + \langle 0|) \tag{51}$$

$$= \frac{3}{4}|1\rangle\langle 1| + \frac{1}{4}|0\rangle\langle 0| + \frac{1}{4}|0\rangle\langle 1| + \frac{1}{4}|1\rangle\langle 0| \tag{52}$$

This is the matrix

$$\frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix} \tag{53}$$

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b) What are the eigenvalues for the density matrix of the second particle and the eigenstates for the density matrix of the second particle.

The eigenvalue equation is

$$\lambda^2 - \lambda + \frac{1}{8} = 0 \tag{54}$$

which has solutions

$$\lambda = (\frac{1}{2} \pm \frac{1}{2}\sqrt{1 - \frac{1}{2}}) = \frac{1}{2}(1 \pm \frac{1}{\sqrt{2}}) \approx .854, .146 \tag{55}$$

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c) What are the eigenvalues for the density matrices for the first particles?

By the Schmidt decomposition, the eigenvalues of the two density matricies are the same.

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