Physics 501-22

Cosmology

We will now look at the Genral Relativity which gives the solution, but the main result is that the universe expands as a function of time. In particular the distance between two nearby objects increases, not because they are moving but because new space is created beteen the objects. If we use x to label the position of ojects at rest, then the distance function between nearby objects is given by (Pythagoras's theorem)

$$ds^2_{space} = a(t)^2(dx^2 + dy^2 + dz^2) = a^2(d\vec{x} \cdot d\vec{x}). \tag{1}$$

Ie, the distance between nearby objects increases as $a(t)$. The special relativitistic spacetime distance is given by

$$ds^2 = dt^2 - ds^2_{space} = dt^2 - a^2(d\vec{x} \cdot d\vec{x}) \tag{2}$$

The equation of motion of a scalar field is

$$\frac{1}{a^3}\partial_t a^3 \partial_t \phi - \frac{1}{a^2}\nabla^2\phi = 0 \tag{3}$$

which can be derived from a Lagrangian

$$\mathcal{L} = \frac{1}{2}\int a^3(\partial_t\phi^2 - \frac{1}{a(t)^2}(\nabla\phi\cdot\nabla\phi))d^3x \tag{4}$$

$a(t)^2 d^3x$ is the volume element of space, given that the spatial distances increase as $a(t)dx$. This is like rd
$theta$ where a little change in the coordinate θ corresponds to an actual physical distance of $rd\theta$.

The conjugate momentum to $\phi(t,x)$ is o

$$\frac{\delta\mathcal{L}}{\delta\partial_t\phi(t,x)} = \pi \tag{5}$$

or

$$\pi = a^3\partial_t\phi \tag{6}$$

and the Hamiltonian is

$$H = \int \pi\partial_t\phi d^3x - \mathcal{L} = \frac{1}{2}(\frac{\pi^2}{a^3} + a|\nabla\phi|^2) \tag{7}$$

The Hamiltonian action is

$$S = \int \pi\partial_t\phi d^3x - H = \int \left[\pi\partial_t\phi - \frac{1}{2}\int(\frac{\pi^2}{a^3} + a\nabla\phi\cdot\nabla\phi\right] d^3x \tag{8}$$

The equations of motion are

$$\partial_t\phi = \frac{\pi}{a^3} \tag{9}$$

$$\partial_t\pi = a\nabla^2\phi \tag{10}$$

We can write the spatial part of this in terms of exponentials of spatial coordinates

$$\phi_k(t,x) = \frac{1}{\sqrt{2\pi}^3}\phi_k(t)\frac{e^{i(k\cdot x)}}{\sqrt{(2\pi)^3}}d^3x \tag{11}$$

and similarly for π_k, with the time dependent equations

$$\partial_t\phi_k = \frac{\pi_k}{a^3} \tag{12}$$
$$\partial_t\pi_k = ak^2\phi_k(t) \tag{13}$$

which come from a Hamilatonian action

$$\frac{H_k = \frac{1}{2}(\pi_k^2}{a^3 + ak^2\phi_k^2)} \tag{14}$$

Thus we have the action for each $\vec{k}$,

$$S_k = \int \pi_k(\partial_t\phi_k) - \frac{1}{2}(\frac{\pi_k^2}{a^3} + ak^2\phi_k^2)d^3kdt \tag{15}$$
$$= \int \pi_k(\partial_t\phi_k) - \frac{1}{2}\frac{k}{a}(\frac{\pi^2}{ka^2} + ka^2\phi_k^2)dtd^3k \tag{16}$$

Comparing this for each k to the expression for the adiabatic expansion we find that

$$\tau_k = \int \frac{k}{a}dt \tag{17}$$
$$\Omega_k = ka^2 \tag{18}$$

We thus have

$$\hat{\pi}_k = \frac{\pi_k}{\sqrt{k}a} - \frac{\dot{a}}{a}\sqrt{k}a\phi_k \tag{19}$$
$$\hat{\phi}_k = \phi_k\sqrt{k}a \tag{20}$$

where $\dot{} = \frac{d}{d\tau_k}$ and

$$\hat{H}_k = \frac{1}{2}(\hat{\pi}_k^2 + \hat{\phi}_k^2(1 - \frac{\ddot{a}}{a})) \tag{21}$$

Now this τ_k depends on k and scales as k for large k.i so $\ddot{a}/a$ will scale as $\frac{1}{k^2}$ and becomes very small for large k. On the other hand for small k this will be very large, and if $\ddot{a} > 0$, $1 - \ddot{a}/a$ will go negative. In that case the solution to the equations of motion will grow or decrease exponentially, with faster growth for smaller k in terms of τ_k.

The other important relation is between the $\hat{}$ momentum and configurtion and the original.

$$\hat{\pi}_k = \pi_k/(\sqrt{k}a) + \dot{a}\frac{\phi_k}{\sqrt{k}} \tag{22}$$
$$\hat{\phi}_k = \sqrt{k}a\phi_k \tag{23}$$

Let us assume that we are looking at large enough k that that the k dependence in $\hat{H}$ can be neglected. Then the solution for $\hat{\phi}$, $\hat{\pi}$ is

$$\hat{\phi}_k = \hat{\phi}_k(0)\cos(k\hat{\tau}) + \hat{\pi}_k(0)\sin(k\hat{\tau}) \tag{24}$$

$$\hat{\pi}_k = \hat{\pi}_k(0)\cos(k\hat{\tau}) + \hat{\phi}_k(0)\sin(k\hat{\tau}) \tag{25}$$

If $a(\tau)$ is exponential, then $\ddot{a}/a$ is constant, and the solution is exact. Since $tau = \int \frac{dt}{a}$ or, $dt = \int a(\tau)d\tau$, if $a(\tau)$ is exponential, $a(t)$ must be linear in t. Ie, for a linearly growing universe, one can solve the equation exactly.

Quantization:

Let us now quantize the field. Defining $\hat{\tau} = \int \frac{k}{a(t)} dtdt$ and write $\hat{a}(\hat{\tau}) = a(t(\hat{\tau}))$.

Let us first define the quantum fields $\Phi(t,x)$ and $\Pi(t,x)$ which obey

$$[\Phi(t,x), \Pi(t,x)] = i\delta^3(x-x') \tag{26}$$

These obey the equations

$$\partial_t \Phi(t,x) = \frac{1}{a(t)^3}\Pi(t,x) \tag{27}$$

$$\partial_t \Pi(t,x) = a(t)\nabla^2\Phi(t,x) \tag{28}$$

Let us now look at the evolution for a very small time δt

$$\Phi(t+\delta t, x) \approx \Phi(t,x) + \delta t \frac{1}{a(t)^3}\Pi(t,x) \tag{29}$$

$$\Pi(t+\delta t, x) \approx \Pi(t,x) + a(t)\nabla^2\Phi(t,x) \tag{30}$$

Now let us take the Hamiltonian diagonalisation plane wave modes, which are defined at time t, so that

$$-i\omega_k \phi_i D\vec{k}(t)\frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{(2\pi)^3}} = \frac{\pi_{D\vec{k}}(t)}{a(t)^3}\frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{(2\pi)^3}} \tag{31}$$

$$-i\omega_k \pi(t)_{D\vec{k}}(t)\frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{(2\pi)^3}} = -k^2 a(t)\phi(t)_{D\vec{k}}\frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{(2\pi)^3}} \tag{32}$$

where the D stands for Diagonalisation. Normalising the modes with the harmonic norm $<\phi', \phi>= \frac{i}{2}\int(\phi'^*(t,x)\pi(t,x) - \pi'^*\phi(t,x))d^3x$ we have

$$\omega_k^2 = \frac{k^2}{a(t)^2} \tag{33}$$

$$\pi_{D\vec{k}}(t) = ika(t)^2\phi_{Dk}(t) \tag{34}$$

Normalizing these modes we get

$$|\phi_{D\vec{k}}|(ka(t)^2) = 1 \tag{35}$$

$$\phi_{D\vec{k}}(t) = \frac{1}{\sqrt{k}a(t)} \tag{36}$$

$$\pi_{D\vec{k}}(t) = -i\sqrt{k}a(t) \tag{37}$$

Thus, the diagonalisation mode at time $t+\delta t$ is

$$\phi_{D\vec{k}}(t+\delta t) = \frac{1}{\sqrt{k}a(t+\delta t)} \approx \phi_{D\vec{k}a}(1-\frac{\partial_t a(t)}{a(t)}\delta t) \tag{38}$$

$$\pi_{D\vec{k}}(t+\delta t) = \pi_{D\vec{k}}(t)(1+\frac{\partial_t a(t)}{a(t)}\delta t) \tag{39}$$

But

$$-(\frac{\partial_t a(t)}{a(t)}\delta t)\phi_{D\vec{k}}\frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{(2\pi)^3}} = -\frac{\partial_t a(t)}{a(t)}\delta t\left(\phi_{D-\vec{k}}\frac{e^{-i\vec{k}\cdot\vec{x}}}{\sqrt{(2\pi)^3}}\right)^* \tag{40}$$

$$(\frac{\partial_t a(t)}{a(t)}\delta t)\pi_{D\vec{k}}\frac{e^{i\vec{k}\cdot\vec{x}}}{\sqrt{(2\pi)^3}} = -\frac{\partial_t a(t)}{a(t)}\delta t\left(\pi_{D-\vec{k}}\frac{e^{-i\vec{k}\cdot\vec{x}}}{\sqrt{(2\pi)^3}}\right)^* \tag{41}$$

Ie, the change in the diagonalisation mode is just proportional to the complex conjugate of the mode for $-\vec{k}$. Ie, it is a mode with negative norm.

Now,

$$A_{D\vec{k}}(t+\delta t) = A_{D\vec{k}} - (\frac{\partial_t a(t)}{a(t)}\delta t)A^\dagger_{D-\vec{k}} \tag{42}$$

Ie, the vacuum of the Hamiltonian diagonalization at time $(t+\delta t)$ will be a many particle state of the Hamiltonian diagonalisation at time t.

$$\int(\langle 0|_{Dt} A^\dagger_{D\vec{k}}(t+\delta t)A_{D\vec{k}}(t+\delta t)\,|0\rangle_{Dt}\, d^3k \tag{43}$$

$$= (\frac{\partial_t a(t)}{a}\delta t)^2\int\langle 0|_{Dt}\, A_{D-\vec{k}}(t)A^\dagger_{D-\vec{k}}(t)\,|0\rangle_{Dt}\, d^3k \tag{44}$$

$$= (\frac{\partial_t a(t)}{a}\delta t)^2\int d^3k \tag{45}$$

Ie, each mode contributes the same number of particles in that small time interval. The total particle creation over the infinitessimal time interval δt is therefor infinite. The vacuum state according the Hamiltonian diagonalisation at time t contains an infinite number of particles as defined by the Hamiltonian diagonaliation at time $t+\delta t$ no matter how small δt is.

This is clearly the wrong answer.

The effective Hamiltonian is

$$H_k = \frac{1}{2}(\frac{1}{a^3}\pi^2_{\vec{k}} + k^2 a\phi^2_{\vec{k}}) \tag{46}$$

Let us make the assymptotic transformations to the $\hat{H}$ variables and Hamiltonian, and time

$$\hat{H}_{\vec{k}} = \frac{1}{2}(\hat{\pi}^2_{\vec{k}} + (1-\frac{\partial^2_{\tau_k}a(\tau_k)}{a(\tau_k)})\hat{\phi}^2_{\vec{k}}) \tag{47}$$

where

$$\tau_k = \int \frac{k}{a(t)} dt \tag{48}$$

$$a(\tau_k) = a(t(\tau_k)) \tag{49}$$

$$\hat{\phi}_{\vec{k}}(t) = \sqrt{k} a(\tau_k) \phi_{\vec{k}} \tag{50}$$

$$\hat{\pi}_{\vec{k}} = \frac{1}{\sqrt{k} a(\tau_k)} \pi_{\vec{k}} + \frac{\dot{a}}{a} \hat{\phi}_{\vec{k}} \tag{51}$$

We now diagonalize this Hamiltonian.

$$-i\hat{\omega}_k \phi_{\vec{k}}(\tau_k) = \hat{\pi}_{\vec{k}}(\tau_k) \tag{52}$$

$$-i\hat{\omega}_k \pi_{\vec{k}}(\tau_k) = -1 - \frac{\ddot{a}(\tau_k)}{a(\tau_k)} \phi_{\vec{k}}(\tau_k) \tag{53}$$

(where again the τ_k dependence is not that of solution to the equations of motion, but the $\hat{H}$ diagonalisation at time τ_k so

$$\hat{\omega}_k^2 = -(1 - \frac{\ddot{a}}{a}) \tag{54}$$

$$\hat{\pi}_{\vec{k}}(\tau_k) = -i\sqrt{(1 - \frac{\ddot{a}(\tau_k)}{a(\tau_k)}} \hat{\phi}_{\vec{k}}(\tau_k) \tag{55}$$

The norm is

$$\frac{i}{2}(\phi_{\vec{k}}(\tau_k)^* \pi_{\vec{k}}(\tau_k) - \pi_{\vec{k}}(\tau_k)^* \phi_{\vec{k}}(\tau_k) = \sqrt{(1 - \frac{\ddot{a}(\tau_k)}{a(\tau_k)}} \tag{56}$$

$$\hat{\phi}_{\vec{k}}(\tau_k) = (1 - \frac{\ddot{a}(\tau_k)}{a(\tau_k)})^{-\frac{1}{4}} \tag{57}$$

$$\hat{\pi}_{\vec{k}}(\tau_k) = -i(1 - \frac{\ddot{a}(\tau_k)}{a(\tau_k)})^{+\frac{1}{4}} \tag{58}$$

$\phi_{\vec{k}}$ is real, and $\pi_{\vec{k}}$ is imaginary and thus must equal $\frac{-i}{\phi_{\vec{k}}}$ to be a normalized mode. Then

$$\hat{\phi}_{\vec{k}}(\tau_k + \delta\tau_k) = \hat{\phi}_{\vec{k}}(\tau)(1 + \frac{\dot{\hat{\phi}}_{\vec{k}}}{\hat{\phi}_{\vec{k}}} \delta\tau_k \tag{59}$$

$$\frac{\hat{\pi}_{\vec{k}} = \hat{\pi}_{\vec{k}}(1 - \dot{\hat{\phi}}_{\vec{k}}}{\phi_{\vec{k}} \delta\tau_k)} \tag{60}$$

Thus we have

$$\hat{\phi}_{\vec{k}}(\tau_k + \delta\tau_k) = \hat{\phi}_{\vec{k}}(1 + \partial_{\tau_k} ln(\phi_{\vec{k}}) \delta\tau_k) \tag{61}$$

$$= \hat{\phi}_{\vec{k}}(1 + \partial_{\tau_k} ln(\phi_{\vec{k}}) k a \delta t) \tag{62}$$

$$\hat{\pi}_{\vec{k}}(\tau_k + \delta\tau_k) = \pi_{\vec{k}}(\tau_k)(1 - ln(\phi_{\vec{k}}) k a \delta t) \tag{63}$$

Again the change (proprtional δt is the complex conjugate of the original. Thus this part of the term will result in a Boguliubov transformation whith $A_{\vec{k}}(t+\delta t)$ being a combination of the annihilation and creation operators at time $t+\delta t$. Now however, the time dependent term $\frac{\ddot{a}}{a}$ scales as $1/k^2$, and the extra tau derivative of this scales as $1/k^3$ and the square of this times $ka\delta t$ scales as $1/k^2$. The integral of this squared goes as $\int 1/k^4 d^3k$ is finite. Ie, we have a finite number of particles created if we define particles via the Hamiltonian diagonalisation for the $\hat{H}$ rather than the original H.

Of course $\hat{H}$ is not the real Hamiltonian, or the real energy of the system.

This whole argument, which was given by L Parker (joined later by S Fulling) in the late 1960's and early 1970's raises the troublesome question– what does one mean by particles in quantum field theory in General Relativity?

It is a problem which is still troublesome even now, 50 years later.