3.11 Problems

1. Prove that E as defined in Eq. (58) is indeed conserved for spatially localized field configurations by using the Gauss theorem together with the property of the Killing vector ξ_μ and the symmetry of the energy-momentum tensor $T_{\mu\nu} = T_{\nu\mu}$.
2. Specify Eqs. (99), (100), (101), (102), and (56) for the Minkowski space-time and express them in terms of the electric and magnetic fields $\boldsymbol{E}$ and $\boldsymbol{B}$ in order to recover the well-known equalities for the Lagrangian density $\mathcal{L} = (\boldsymbol{E}^2 - \boldsymbol{B}^2)/2$; the vector potentials $\boldsymbol{B} = -\boldsymbol{\nabla} \times \boldsymbol{A}$ and $\boldsymbol{E} = \dot{\boldsymbol{A}} + \boldsymbol{\nabla}\phi$, which automatically satisfy the two source-free Maxwell equations for the vacuum $\boldsymbol{\nabla} \cdot \boldsymbol{E} = 0$ and $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$; the remaining dynamical Maxwell equations $\boldsymbol{\nabla} \times \boldsymbol{B} = \dot{\boldsymbol{E}}$ and $\boldsymbol{\nabla} \times \boldsymbol{E} = -\dot{\boldsymbol{B}}$; the energy density $\mathcal{E} = (\boldsymbol{E}^2 + \boldsymbol{B}^2)/2$; as well as the Poynting theorem $\dot{\mathcal{E}} + \boldsymbol{\nabla} \cdot \boldsymbol{S} = 0$ with $\boldsymbol{S} = \boldsymbol{E} \times \boldsymbol{B}$ being Poynting vector.
3. Show that the solutions in Eq. (93) are indeed quasi-orthogonal.

4 Quantum fields in curved space-time

After having discussed some aspects of classical fields in curved space-times, we now go on and turn our attention to the quantum effects. For reasons of simplicity, we start with the free, minimally coupled, massless, and neutral scalar field. The main ideas can by applied to other free fields (e.g., Dirac fermions or photons) as well – some of these generalisations will be discussed at the end of this Chapter.

4.1 Simple example: single harmonic oscillator

Before turning our attention to the quantum field, let us repeat the basic quantum theory of a harmonic oscillator, which already yields fruitful insight. In the static situation, the Hamiltonian is given by

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \Omega^2\,\hat{q}^2\right)\,, \tag{183}$$

with a constant frequency Ω and the canonical commutation relations $[\hat{q}, \hat{p}] = i$. (The mass as well as Planck's constant $\hbar$ are set to one.) The creation $\hat{a}^\dagger$ and annihilation $\hat{a}$ operators defined via $\hat{a} = (\Omega\hat{q} + i\hat{p})/\sqrt{2\Omega}$ obey the commutation relation $[\hat{a}, \hat{a}^\dagger] = 1$. This identity is sufficient for deriving the structure of the Hilbert space and the spectrum, etc.; for illustration we shall present a short repetition of the main arguments:

- The operator $\hat{n} = \hat{a}^\dagger\hat{a}$ is evidently non-negative as well as self-adjoint and thus possesses a complete set of eigenvectors $\hat{n}\,|n\rangle = n\,|n\rangle$ with eigenvalues $n \geq 0$.

- With $[\hat{a}, \hat{a}^\dagger] = 1$ it follows $[\hat{n}, \hat{a}] = -\hat{a}$ and $[\hat{n}, \hat{a}^\dagger] = \hat{a}^\dagger$. Acting these operator identities on an eigenvector $|n\rangle$ one obtains $\hat{a}\,|n\rangle \propto |n-1\rangle$ and $\hat{a}^\dagger\,|n\rangle \propto |n+1\rangle$, which leads to the interpretation of $\hat{a}^\dagger$ and $\hat{a}$ as ladder operators.

- Sandwiching the above operator identities between the states $|n\rangle$ and $|n \pm 1\rangle$ and using the fact that all states are normalised, we deduce the pre-factors $\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle$ and $\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle$.

- Since the operator $\hat{n}$ is non-negative $n \geq 0$ the multiple application of the lowering ladder operator $\hat{a}$ on an eigenvector $|n\rangle$ has to yield zero eventually and hence we may conclude $n \in \mathbb{N}$.

The Hilbert space of every quantum system can be completely determined by simultaneous diagonalisation of a maximum number of independent and commuting operators/observables. In this static and one-dimensional system, there exists only one conserved quantity – the Hamiltonian (i.e., the energy) itself[13].
Indeed, there are no further independent and commuting operators and the Hamiltonian is related to the operator $\hat{n}$ via $\hat{H} = \Omega(\hat{n} + 1/2)$. The second addend 1/2 describes the zero-point energy of the ground-state fluctuations as demanded by the Heisenberg uncertainty principle. Motivated by the integer spectrum of $\hat{n}$ and its relation to the Hamiltonian and the energy, one may assign a particle interpretation, where $\hat{a}^\dagger$ and $\hat{a}$ are to be identified with creation and annihilation operators.
Note that one may define new operators as linear combinations $\hat{a}' = \alpha\hat{a} + \beta\hat{a}^\dagger$ of the original ones. If $|\alpha|^2 - |\beta|^2 = 1$ holds then the new operators do also satisfy the same commutation as the old ones relation and hence could also be interpreted as ladder operators. (As it will become evident shortly, this is also called a Bogoliubov transformation.) However, they do not diagonalise the Hamiltonian – i.e., one cannot rewrite the Hamiltonian in terms of the new 'number' operator $\hat{n}'$ – and thus one cannot apply the particle interpretation.
Let us now consider the dynamical situation

$$\hat{H}(t) = \frac{1}{2}\left(\hat{p}^2 + \Omega^2(t)\,\hat{q}^2\right) , \tag{184}$$

where the potential $\Omega^2(t)\,\hat{q}^2$ may change. For this purpose it is convenient to adopt the Heisenberg representation. The operator $\hat{q}(t)$ then satisfies the second-order ordinary

[13] More generally, in classical integrable systems, there are exactly as many independent conserved quantities whose mutual Poisson brackets vanish as degrees of freedom. Non-integrable, i.e., chaotic systems possess even less. In quantum theory, the Poisson brackets have to be replaced by commutators (in the bosonic case).

differential equation $[d^2/dt^2 + \Omega^2(t)]\hat{q}(t) = 0$. For an oscillating potential, for example, we obtain the Mathieu equation and all the related phenomena, such as parametric resonance, instability bands, and Floquet exponents, etc.
The (conserved) Wronskian associated to the second-order ordinary differential equation – for general $\Omega^2(t)$ – reads $W[F] = F^* \dot{F} - \dot{F}^* F$. This conserved quantity can be generalised to a time-independent inner product in analogy to Section 3.4 via

$$(F|F') = i \left(F^* \dot{F}' - \dot{F}^* F' \right) . \tag{185}$$

Evidently it has the same properties as in Section 3.4. This inner product proves very useful for the classification of the solutions – which are no longer given by simple trigonometric expressions. The decomposition of the operator $\hat{q}(t) = \hat{a} \exp\{-i\Omega t\}/\sqrt{2\Omega} + \hat{a}^\dagger \exp\{+i\Omega t\}/\sqrt{2\Omega}$ in the static case now has to be replaced by

$$\hat{q}(t) = \hat{a}\, F(t) + \hat{a}^\dagger\, F^*(t) , \tag{186}$$

with F denoting a quasi-normalised solution $(F|F) = 1$ of the differential equation $\ddot{F}(t) + \Omega^2(t)F(t) = 0$. Vice versa we may obtain the operator $\hat{a}$ by projection $\hat{a} = (F|\hat{q})$ and with the aid of the canonical equal-time commutation relation $[\hat{p}(t), \hat{q}(t)] = \pm i$ we obtain

$$[(F|\hat{q}) , (\hat{q}|F')] = (F|F') . \tag{187}$$

Hence the operators $\hat{a}$ and $\hat{a}^\dagger$ satisfy the same commutation relation $[\hat{a}^\dagger, \hat{a}]$ as in static case – for all quasi-normalised $(F|F) = 1$ solutions F. Note, however, that they do not diagonalise the Hamiltonian $\hat{H}(t)$ for all times t in general.
During every static period $\Omega^2 = \text{const}$, however, there are uniquely defined solutions $F(t)$ with the associated operators $\hat{a} = (F|\hat{q})$ and $\hat{a}^\dagger$ diagonalising the Hamiltonian, namely $F(t) = \exp\{-i\Omega t\}/\sqrt{2\Omega}$ (see the beginning of this Section). Let us assume that the potential $\Omega^2(t)$ of the oscillator does only vary during a finite period of time $t_{\text{in}} < t < t_{\text{out}}$ and is therefore stationary for asymptotic times $\Omega^2(t < t_{\text{in}}) = \Omega^2_{\text{in}}$ and $\Omega^2(t > t_{\text{out}}) = \Omega^2_{\text{out}}$. In this case we may expand the operator $\hat{q}(t)$ into the two sets

$$\hat{q}(t) = \hat{a}_{\text{in}} F_{\text{in}}(t) + \hat{a}^\dagger_{\text{in}} F^*_{\text{in}}(t) = \hat{a}_{\text{out}} F_{\text{out}}(t) + \hat{a}^\dagger_{\text{out}} F^*_{\text{out}}(t) , \tag{188}$$

with $F_{\text{in}}(t < t_{\text{in}}) = \exp\{-i\Omega_{\text{in}} t\}/\sqrt{\Omega_{\text{in}}}$ and $F_{\text{out}}(t > t_{\text{out}}) = \exp\{-i\Omega_{\text{out}} t\}/\sqrt{\Omega_{\text{out}}}$ – which diagonalise the Hamiltonian in the initial $\hat{H}(t < t_{\text{in}}) = \Omega_{\text{in}}(\hat{a}^\dagger_{\text{in}} \hat{a}_{\text{in}} + 1/2)$ and in the final $\hat{H}(t > t_{\text{out}}) = \Omega_{\text{out}}(\hat{a}^\dagger_{\text{out}} \hat{a}_{\text{out}} + 1/2)$ period, respectively.
Note that, via the equation of motion $\ddot{F}(t) + \Omega^2(t)F(t) = 0$, the in-/out-solutions $F_{\text{in}}/F_{\text{out}}$ are defined for all times, but may assume a rather complicated form during as well as

after/before the dynamical period. In complete analogy to Section 3.4 these two sets of solutions may be related to each other via the Bogoliubov coefficients

$$F_{\rm out} = (F_{\rm in}|F_{\rm out})\; F_{\rm in} - (F_{\rm in}^*|F_{\rm out})\; F_{\rm in}^* = \alpha\, F_{\rm in} - \beta\, F_{\rm in}^*\,, \tag{189}$$

which can be derived by means of the inner product defined above. Consequently, the initial and final creation/annihilation operators obey a corresponding relation

$$\hat{a}_{\rm out} = \alpha^*\, \hat{a}_{\rm in} + \beta^*\, \hat{a}_{\rm in}^\dagger\,. \tag{190}$$

The explicite calculation of the Bogoliubov coefficients α and β for a given potential $\Omega^2(t)$ can be mapped to finding the transmission and reflection coefficients in a one-dimensional scattering[14] problem – which is rather complicated in general.
For the sake of simplicity, therefore, we adopt the sudden approximation by assuming a very rapid change of $\Omega^2(t)$ in an extremely short time $\Omega(t) = \Omega_{\rm in}\Theta(-t) + \Omega_{\rm out}\Theta(+t)$. In this simple case the Bogoliubov coefficients can be calculated easily at $t = 0$ (since $\dot{F}$ is still continuous)

$$\beta = (F_{\rm in}^*|F_{\rm out}) = i(F_{\rm in}\,\dot{F}_{\rm out} - \dot{F}_{\rm in}\,F_{\rm out}) = \frac{\Omega_{\rm out} - \Omega_{\rm in}}{2\sqrt{\Omega_{\rm in}\Omega_{\rm out}}}\,. \tag{191}$$

Another way of understanding this scenario is the following: During the sudden change of the potential $\Omega^2(t)$ the wave-function in the position representation $\Psi(q) = \langle q|0_{\rm in}\rangle$ describing the initial ground state $|0_{\rm in}\rangle$ does not have any time to change – and therefore does no longer correspond to the new ground state $|0_{\rm out}\rangle$. Intuitively speaking, the wave-function $\Psi(q) = \langle q|0_{\rm in}\rangle$ is too slim (for $\Omega_{\rm in} < \Omega_{\rm out}$) or to broad (for $\Omega_{\rm in} > \Omega_{\rm out}$) for the new ground state $|0_{\rm out}\rangle$. More exactly, the two states are related via a so-called squeezing transformation

$$|0_{\rm in}\rangle = \hat{S}(\xi)\,|0_{\rm out}\rangle = \exp\left\{\frac{\xi}{2}\left[(\hat{a}_{\rm out}^\dagger)^2 - (\hat{a}_{\rm out})^2\right]\right\}|0_{\rm out}\rangle\;. \tag{192}$$

Evidently the squeezing operator $\hat{S}(\xi)$ is unitary $\hat{S}^\dagger(\xi) = \hat{S}(-\xi) = \hat{S}^{-1}(\xi)$. It can be shown (see the Problems) that it acts as $\hat{S}^\dagger(\xi)\,\hat{q}\,\hat{S}(\xi) = \exp\{-\xi\}\,\hat{q}$ and $\hat{S}^\dagger(\xi)\,\hat{p}\,\hat{S}(\xi) = \exp\{+\xi\}\,\hat{p}$ – and hence changes the variances $\Delta(q) = \sqrt{\langle\hat{q}^2\rangle - \langle\hat{q}\rangle^2} \to \exp\{-\xi\}\,\Delta(q)$ and

[14]For example, assuming an evolution corresponding to a so-called reflection-less potential, such as $\Omega^2(t) = \Omega_0^2 + 2\nu^2/\cosh^2(2\nu t)$, the relection coefficient – and hence also the Bogoliubov β-coefficient – vanish. In most of the situations, however, β will be non-vanishing.

$\Delta(p) \to \exp\{+\xi\}\,\Delta(p)$ maintaining the minimal Heisenberg uncertainty $\Delta(q)\Delta(p) = 1/4$. Thus the Bogoliubov transformation reads

$$\hat{a}_{\rm in} = \hat{S}^\dagger(\xi)\hat{a}_{\rm out}\hat{S}(\xi) = \hat{a}_{\rm out}\cosh\xi + \hat{a}^\dagger_{\rm out}\sinh\xi\,, \tag{193}$$

which enables us to calculate the expectation value of the 'new' number operator $\hat{n}_{\rm out}$ in the 'old' ground state

$$\langle 0_{\rm in}|\,\hat{n}_{\rm out}\,|0_{\rm in}\rangle = |\beta|^2 = \sinh^2\xi\,. \tag{194}$$

After having envisioned the Bogoliubov transformation by means of this simple harmonic-oscillator example, let us dwell a little on the commutation relations: Even though the equal-time commutator $i[\hat{p}(t),\hat{q}(t)] = 1$ is still valid, the general expression for the static oscillator $i[\hat{q}(t),\hat{q}(t')] = \sin(\Omega[t-t'])/\Omega$ obviously no longer holds for varying $\Omega^2(t)$. It is, however, possible to generalise this equality with the aid of the retarded and advanced Green functions defined via

$$\left(\frac{\partial^2}{\partial t^2} + \Omega^2(t)\right)\mathfrak{G}_{\rm ret/adv}(t,t') = \delta(t-t')\,, \tag{195}$$

with $\mathfrak{G}_{\rm ret}(t<t') = 0$ and $\mathfrak{G}_{\rm adv}(t>t') = 0$. Since $\Omega^2(t)$ is assumed to be finite, one can immediately read off the boundary conditions $\mathfrak{G}_{\rm ret/adv}(t=t') = 0$ and $\dot{\mathfrak{G}}_{\rm ret/adv}(t=t') = \pm 1$.

On the other hand, as in every (linear) initial value problem, the operator $\hat{q}(t)$ can be expressed as a linear combination

$$\hat{q}(t) = \hat{q}(t')\mathfrak{A}(t,t') + \hat{p}(t')\mathfrak{B}(t,t')\,, \tag{196}$$

with $\mathfrak{A}(t,t')$ and $\mathfrak{B}(t,t')$ being solutions of the equation of motion which satisfy the boundary conditions $\mathfrak{A}(t=t') = 1$, $\dot{\mathfrak{A}}(t=t') = 0$, $\mathfrak{B}(t=t') = 0$, and $\dot{\mathfrak{B}}(t=t') = 1$. By uniqueness of the solutions, we may identify

$$\mathfrak{B}(t,t') = \mathfrak{G}_{\rm ret}(t,t') - \mathfrak{G}_{\rm adv}(t,t')\,, \tag{197}$$

which leads to the following generalisation of equal-time commutation relation $[\hat{p}(t),\hat{q}(t)] = \pm i$

$$i\,[\hat{q}(t),\hat{q}(t')] = \mathfrak{G}_{\rm ret}(t,t') - \mathfrak{G}_{\rm adv}(t,t')\,. \tag{198}$$

4.2 Scalar field quantisation

In oder to obtain a reasonable notion of the quantum field we first have to impose additional conditions on the structure of the space-time. We assume a strongly causal space-time which enables us to distinguish future and past and hence cause and effect, etc., and forbids the occurrence of closed time-like curves (time-machines), for example. In this case there exist unique (and distinguishable) advanced and retarded Greens functions

$$\Box\,\mathfrak{G}_{\rm ret/adv}(\underline{x},\underline{x}') = \sqrt{-g}\,\delta^4(\underline{x}-\underline{x}')\,. \tag{199}$$

In terms of Greens functions – which are actually not well-defined functions but bi-distributions – we may express the (free-field) canonical commutation relations in a generally covariant form

$$i\left[\hat\Phi(\underline{x}),\hat\Phi(\underline{x}')\right] = \mathfrak{G}_{\rm ret}(\underline{x},\underline{x}') - \mathfrak{G}_{\rm adv}(\underline{x},\underline{x}')\,. \tag{200}$$

Note that we work in the Heisenberg representation where the operators carry all the time-dependence whereas the states $|\Psi\rangle$ do not evolve. The quantum field $\hat\Phi$ is represented by an operator-valued distribution satisfying the wave equation $\Box\hat\Phi = 0$ in the sense of distributions, i.e., for all test functions F it yields $\hat\Phi[\Box F] = 0$.
For example, in flat space-time, the Greens functions (199) have the simple structure $\Theta(\pm\Delta t)\,\delta(\Delta t^2 - \Delta\boldsymbol{r}^2)$ for a free massless scalar field. Note that their support (set of points with non-vanishing values) lies entirely in the future/past light cone – which is no longer the case for massive fields or in general curved space-times (even for massless case). For the Minkowski metric the commutator in Eq. (200) reproduces the equal-time commutation relations in the usual well-known form

$$\begin{aligned}\left[\hat\Phi(t,\boldsymbol{r}),\hat\Phi(t,\boldsymbol{r}')\right] &= \left[\hat\Pi(t,\boldsymbol{r}),\hat\Pi(t,\boldsymbol{r}')\right] = 0\\ \left[\hat\Pi(t,\boldsymbol{r}),\hat\Phi(t,\boldsymbol{r}')\right] &= i\,\delta^3(\boldsymbol{r}-\boldsymbol{r}')\,,\end{aligned} \tag{201}$$

with $\Pi = \delta\mathcal{A}/\delta\dot\Phi$ denoting the canonical momentum density, which in flat space-time is simply given by $\Pi = \dot\Phi$.
The commutator of the inner products of the field $\hat\Phi$ with two (complex) solutions F and F' of the Klein-Fock-Gordon equation turns out to give

$$\left[\left(F|\hat\Phi\right),\left(\hat\Phi|F'\right)\right] = (F|F')\ . \tag{202}$$

This can easily be verified for the simple flat space-time example in Eq. (201) and with a similar slicing Σ for general metrics.

Consequently, the inner products of the field $\hat{\Phi}$ with quasi-orthogonal positive pseudo-norm functions F_I as in Eq. (83)

$$\hat{a}_I = \left(F_I|\hat{\Phi}\right) \tag{203}$$

obey the commutation relations

$$\begin{aligned} \left[\hat{a}_I, \hat{a}_J\right] &= \left[\hat{a}_I^\dagger, \hat{a}_J^\dagger\right] = 0 \\ \left[\hat{a}_I^\dagger, \hat{a}_J\right] &= \delta(I, J)\,. \end{aligned} \tag{204}$$

If we assume the index I to be discrete $I \in \mathbb{N}$ (see the comment below) the operators $\hat{a}_I^\dagger$ and $\hat{a}_I$ exactly describe the creation and annihilation operators of (a set of) independent harmonic oscillators with the number operator $\hat{n}_I = \hat{a}_I^\dagger \hat{a}_I$. Therefore, the Fock space $\mathfrak{F}$ containing all possible states of the quantum field can be written as (the completion of) a direct (orthogonal) product of an infinite set of Hilbert spaces $\mathfrak{H}$ corresponding to (independent) harmonic oscillators

$$\mathfrak{F} = \overline{\bigotimes_I} \mathfrak{H}_I\,. \tag{205}$$

This representation is complete since we may use the completeness of the set of solutions $\{F_I, F_I^*\}$ in order to expand the field $\hat{\Phi}$ via

$$\hat{\Phi}(\underline{x}) = \sum\!\!\!\!\!\!\!\int_I \left(\hat{a}_I\, F_I(\underline{x}) + \hat{a}_I^\dagger\, F_I^*(\underline{x})\right)\,, \tag{206}$$

which enables us to specify the action of $\hat{\Phi}$ on an arbitrary state $|\Psi\rangle \in \mathfrak{F}$.
Note that the above procedure is well-defined for discrete indices I only. For instance, if one chooses the plane waves in Eq. (88) then the index I would correspond to the wavenumber $\boldsymbol{k}$ and hence be continuous $I \simeq \boldsymbol{k} \in \mathbb{R}^3$ for an infinite spatial volume. In this case the r.h.s. of Eq. (204) is ill-defined for coinciding arguments $\delta(I, I) = \infty$ which corresponds to the infinite-volume divergence [...] and complicates the analysis. Also the construction (205) of the Fock space $\mathfrak{F}$ itself is not completely independent of the particular choice of the set $\{F_I\}$ for an infinite volume in the sense that one can find a finite unitary transformation mediating between the two because the scalar product of different states is not always finite.
Nevertheless, one may use normalised wave-packets instead of plane waves as a set of solutions $\{F_I\}$ which thus can be labelled by a discrete index and still approximate the

plane waves locally to an arbitrary accuracy. Even though the Fock space $\mathfrak{F}$ is still not unique in this case for an infinite volume, the above analysis is well-defined. Another possibility would be to assume a finite volume V and consider the limit $V \to \infty$ afterwards, see the comments in the next Section.

4.3 Particle and vacuum definition via energy

The commutation relations in Eq. (204) which correspond to harmonic oscillators seem already to suggest a vacuum definition via the associated ground state(s). However, we have to bear in mind that the decomposition (206) – and thereby this naïve vacuum definition as well – crucially depend[15] on the choice of the basis set $\{F_I\}$.
Since a reasonable particle and thus vacuum definition should not depend on such an ad hoc selection, one has to impose appropriate conditions which have to be satisfied. Let us characterise a reasonable notion of particles by three features:

- Countability
  We wish to be able to count the particles, i.e., no particle (vacuum), one, two etc. – but never 3/2 particles, for example. However, a superposition of, e.g., single-particle and 2-particle states is possible.

- Independence
  For the free field case (which we are considering only[16]) each particle should not be affected by the presence of the others.

- Energy
  Last but not least we expect every particle to carry a definite amount of energy.

Whereas the first two properties already follow from Eq. (204) the third requirement is sharp enough to distinguish the different sets of solutions. As we have observed in Sec. 3.1, the introduction of a conserved energy is possible in stationary space-times only. Moreover, it refers explicitly to the associated time-like Killing vector, cf. Eqs. (62) and (63), which describes the time-evolution of the corresponding observer. For the scalar field under consideration it reads, cf. Sec. 3.1

$$E = \frac{1}{2} \int d^3r \left(g^{00}\, \dot{\Phi}^2 - g^{ij} (\partial_i \Phi)(\partial_j \Phi) \right) . \tag{207}$$

[15]This will become even more evident in the next Section, where the phenomenon of particle creation is discussed.

[16]In the case of interacting quantum fields the notion of particles is much more complicated. For example, in Quantum electro-dynamics (QED) it is hard – if not impossible – to separate electrons cleanly from photons.

Quite reasonably, the integrand equals the Hamiltonian density $\mathcal{H} = \Pi\dot{\Phi} - \mathcal{L}$.
In the following considerations we shall assume a static space-time for simplicity, for general stationary metrics the derivations are more complicated but lead to the same results. For a block-diagonal metric $g_{0i} = 0$ the KFG equation assumes the form

$$\ddot{\Phi} = -\frac{g_{00}}{\sqrt{-g}}\,\partial_i\left(\sqrt{-g}\,g^{ij}\,\partial_j\Phi\right) \stackrel{\text{def}}{=} -\mathcal{K}\Phi\,, \tag{208}$$

where we have introduced the second-oder differential operator $\mathcal{K}$ containing the spatial derivatives.
Since the metric is static $g_{0i} = g^{0i} = 0$ and the Killing vector $\partial/\partial t$ is time-like, i.e., $g_{00} = 1/g^{00} > 0$, the spatial metric g^{ij} has to be negative definite (strictly speaking, its eigenvalues). The existence of two time-like space-time directions would entail the possibility of closed time-like curves – which is excluded by causality. Therefore the signature of the metric must be non-degenerated and hence $\mathcal{K}$ is non-negative.
In addition, if the space-time is physically complete, the operator is self-adjoint $\mathcal{K} = \mathcal{K}^\dagger$ with respect to scalar product with measure $d^3r\,g^{00} = d^3x\,\sqrt{-g}\,g^{00}$. Here physical completeness demands that the surface terms is a spatial integration by parts vanish, i.e., that the spatial hyper-surfaces have nor edges or holes without appropriate boundary conditions (such as Dirichlet type for perfect reflection). A strongly causal and complete space-time is also called globally hyperbolic [...].
By virtue of the spectral theorem every self-adjoint operator can be diagonalised by a suitable unitary transformation $\mathcal{U}$, i.e., $[\mathcal{U}^\dagger\,\mathcal{K}\mathcal{U}\,f](\lambda) = \lambda\,f(\lambda)$ for all functions f defined on the spectrum of $\mathcal{K}$. Formally, one may use this unitary mapping $\mathcal{U}$ – which is the generalisation of the Fourier transformation $\mathcal{F}$ for curved space-times – in order to construct a complete set of quasi-eigenfunctions f_I via $f_I = \mathcal{U}\,\delta(\lambda, \lambda_I)$ which then implies $\mathcal{K}\,f_I = \lambda_I\,f_I$.
However, depending on the concrete character of the spectrum of $\mathcal{K}$, the so defined quasi-eigenfunctions f_I may have rather nasty properties (singular continuous spectrum, fractal measure, etc.). Fortunately, assuming a sufficiently well-behaving space-time we obtain either proper eigenfunctions for a finite spatial volume (discrete spectrum) or generalised eigenfunctions for an infinite volume (absolute continuous spectrum).
Having at hand those eigenfunctions we may construct a complete set of solutions F_I of the wave equation via the positive pseudo-norm functions

$$F_I(t, \boldsymbol{r}) = \exp\left\{-i\omega_I t\right\}\,f_I(\boldsymbol{r})\,, \tag{209}$$

and their complex conjugates (negative pseudo-norm). Expansion of the field $\hat{\Phi}$ into these functions according to Eq. (206) and insertion into the energy (and Hamiltonian) in Eq. (207) finally yields

$$\hat{E} = \sum\!\!\!\!\!\!\!\int_I \frac{\omega_I}{2} \left(\hat{a}_I^\dagger \hat{a}_I + \hat{a}_I \hat{a}_I^\dagger \right) = \sum\!\!\!\!\!\!\!\int_I \omega_I \, \hat{a}_I^\dagger \hat{a}_I + E_{\rm zero-point} \,. \tag{210}$$

Evidently this energy-operator possesses a unique ground state which serves as a physically reasonable vacuum definition – at least in the absence of any horizons, etc. (as we shall see later)

$$\forall_I \; \hat{a}_I \, |0\rangle = 0 \,. \tag{211}$$

Note that the above expressions for the energy and thus also for the associated ground state explicitly refer to a particular Killing vector ξ_μ describing the time-coordinate t of a special observer and hence do not represent a covariant concept.
The non-vanishing ground-state energy $E_{\rm zero-point}$ in Eq. (210) is infinite in complete analogy to flat space-time. But in curved space-times it does depend on the geometry and topology in general which gives raise to Casimir-type effects, e.g., for a closed universe $\mathbb{M} = \mathbb{R} \otimes \mathbb{S}_3$.
As already discussed in the previous Section, in the case of an infinite spatial volume leading to an absolute continuous spectrum, one may utilise wave-packets instead of plane waves in order to construct a complete and quasi-orthogonal set of positive/negative pseudo-norm solutions F_I of the wave equation. Although these function would not diagonalise the energy operator exactly, they can be chosen in such a way that they diagonalise it approximately in the vicinity of an observer – which for all practical purposes should be sufficient.
Similarly, if the space-time does not admit an exact global (time-like) Killing vector – but only a locally approximate one, which describes the time of some observer, a similar approximate diagonalisation and thus particle definition can be accomplished.

4.4 Particle and vacuum definition via detectors

If the space-time does not possess (an approximate) time-like Killing vector, one has to find an alternative method for defining the notion of particles. This can be accomplished by considering the response of a class of particle detectors – whose dynamics then fix the choice for the field modes F_I.
Let us first consider one single particle detector at rest in flat space-time and assume for the sake of simplicity that it can be described by a two-level system ("on" or "off") with

the Hamiltonian

$$H = \frac{\Omega}{2}\,\sigma_z + \lambda\Phi\,\sigma_x\,, \tag{212}$$

where Ω is the energy gap between the two levels, λ the (level-transition) coupling constant to the field Φ, and $\sigma_x, \sigma_y, \sigma_z$ denote the usual Pauli spin-matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,\ \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,,\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{213}$$

Regarding $H_0 = \Omega\sigma_z$ as the undisturbed Hamiltonian and switching to the interaction picture we obtain

$$H_{\rm int}(t) = \lambda\Phi(t)\left(\sigma_+ e^{+i\Omega t} + \sigma_- e^{-i\Omega t}\right)\,, \tag{214}$$

with $\sigma_\pm = (\sigma_x \pm i\sigma_y)/2$ and hence $\sigma_+^\dagger = \sigma_-$.
Within the rotating wave approximation (i.e., for large measurement times), we may neglect all terms with residual oscillations and therefore we arrive at

$$H_{\rm int}^{\rm RWA} = \lambda\left(\widetilde{\Phi}(+\Omega)\,\sigma_+ + \widetilde{\Phi}(-\Omega)\,\sigma_-\right)\,. \tag{215}$$

Let us assume that the detector is originally in the ground state of H_0 (which corresponds to "off")

$$\left|\Psi^0_{\rm detector}\right\rangle = |\downarrow\rangle\,, \tag{216}$$

where we have adopted the spin-like representation

$$\begin{array}{rclcrcl} \sigma_z\,|\uparrow\rangle & = & +\,|\uparrow\rangle & , & \sigma_z\,|\downarrow\rangle & = & -\,|\downarrow\rangle \\ \sigma_+\,|\uparrow\rangle & = & 0 & , & \sigma_+\,|\downarrow\rangle & = & |\uparrow\rangle \\ \sigma_-\,|\uparrow\rangle & = & |\downarrow\rangle & , & \sigma_-\,|\downarrow\rangle & = & 0 \end{array}\,, \tag{217}$$

i.e., $\sigma_+ = |\uparrow\rangle\,\langle\downarrow|$.
With the usual flat space-time decomposition of the quantum field $\hat{\Phi}$ its interaction with the detector is governed by the so-called Jaynes-Cunnings Hamiltonian

$$\hat{H}_{\rm int}^{\rm RWA} = \lambda\left(\hat{a}_\Omega\,\sigma_+ + \hat{a}_\Omega^\dagger\,\sigma_-\right)\,. \tag{218}$$

Consequently, if the quantum state of the field $\hat{\Phi}$ is the Minkowski vacuum $|0\rangle$ there is no response of the particle detector – i.e., its state $|\downarrow\rangle$ ("off") does not change – as one would expect

$$\hat{H}_{\rm int}^{\rm RWA}\,|0\rangle\,|\downarrow\rangle = 0\,. \tag{219}$$

Now let us consider a detector which is moving in an arbitrary curved space-time along a trajectory $\underline{x}[\tau]$ with τ being the proper time. If we decompose the field operator

$$\hat{\Phi}(\underline{x}) = \sum\!\!\!\!\!\!\int_I \hat{a}_I \, F_I(\underline{x}) + \text{h.c.} \,, \tag{220}$$

into modes $F_I(\underline{x})$ which behave along the world-line of the detector $\underline{x}[\tau]$ as

$$F_I\left(\underline{x}[\tau]\right) \propto \exp\left\{-i\omega_I \, \tau\right\} \,, \tag{221}$$

then the vacuum state defined via $\forall_I \, \hat{a}_I \, |0\rangle = 0$ again does not trigger a response of the detector (for late times, i.e., in the rotating wave approximation).
In this way one can select a set of modes F_I and thereby a particle and vacuum definition via Eq. (221) by means of a sufficiently large set of detectors (with space-time filling trajectories and all frequencies) instead of the one based on a Killing vector – provided that all detectors "agree" on the same vacuum state.

4.5 Particle creation – Bogoliubov coefficients

As it became evident in the previous Sections, the notion of particles and the vacuum state is neither covariant nor unique and depends on the particular Killing vector or observer. Now we are going to examine the consequences of possible deviations. Let us assume that there are two different observers O and O' whose time evolution t and t' corresponds to non-coinciding Killing vectors ξ_μ and ξ'_μ.
Note that these observers (and Killing vectors) do not need to coexist at the same space-time location. Although the associated complete sets of solutions F_I and F'_I can be uniquely extended (initial value problem) to the whole space-time, the Killing vectors ξ_μ and ξ'_μ may exist in particular space-time regions only.
Clearly, expanding the field $\hat{\Phi}$ into different sets of solutions

$$\hat{\Phi}(\underline{x}) = \sum\!\!\!\!\!\!\int_I \hat{a}_I \, F_I(\underline{x}) + \text{h.c.} = \sum\!\!\!\!\!\!\int_J \hat{a}'_J \, F'_J(\underline{x}) + \text{h.c.} \,, \tag{222}$$

the two distinct observers O and O' will define different sets of creation and annihilation operators $\hat{a}_I$ and $\hat{a}'_I$ as well as non-coinciding vacuum states $|0\rangle$ and $|0'\rangle$ in general. The relation between the two sets can be expressed with the aid of the Bogoliubov coefficients introduced in Sec. 3.4

$$\hat{a}'_J = \sum\!\!\!\!\!\!\!\int_I \left(\alpha^*_{IJ}\hat{a}_I + \beta^*_{IJ}\hat{a}^\dagger_I\right) . \tag{223}$$

The Bogoliubov β-coefficients describe the mixing of positive and negative pseudo-norm solutions in the two sets $\{F_I\}$ and $\{F'_I\}$. Evidently, the two vacua $|0\rangle$ and $|0'\rangle$ do only coincide if all β-coefficients vanish. Indeed, the above expression enables us to derive the following very important relation describing the content of particles $\hat{n}'_J$ as seen by the one observer O' within the vacuum $|0\rangle$ defined by the other one O

$$\langle 0|\,\hat{n}'_J\,|0\rangle = \sum\!\!\!\!\!\!\!\int_I |\beta_{IJ}|^2 . \tag{224}$$

As an illustrative example one might consider a space-time which is for asymptotic times ($t \uparrow +\infty$ and $t \downarrow -\infty$) stationary and undergoes a dynamical period in between. In this case it is not possible to accomplish a particle definition which is valid throughout but merely in the two asymptotic-time regions. Note that the metric does not necessarily coincide before $t \uparrow +\infty$ and after $t \downarrow -\infty$ the dynamical phase. If an (in) observer O finds the quantum field initially in its ground state $|0\rangle$ Eq. (224) tells us that after the dynamical period this is no longer true and a later (out) observer O' will detect particles in general. (We are working in the Heisenberg picture.) In the common (and reasonable) way this result is interpreted as that these particles are created by the interaction with the time-dependent gravitational field – remember the second term in Eq. (56).
In the example described above the two observers O and O' did not coexist. However, as we shall see in the following Section, one might obtain non-trivial effects even if the world-lines of the two detectors intersect.

4.6 Unruh effect

In order to illustrate the significance of the observer with respect to the particle concept we shall discuss a simple example where the vacuum of one observer does not appear to be empty (i.e., free of particles) to another one.
Let us consider a massless and non-interacting neutral scalar field in a 3+1 dimensional flat space-time. The two-point Wightman [...] function of the Minkowski vacuum $|0\rangle$

(ground state of all inertial observers) is uniquely determined by the Wightman axioms and the scale invariance (for a massless field). Away from the light-cone[17] it simply reads

$$\langle 0| \hat{\Phi}(\underline{x})\hat{\Phi}(\underline{x}')\,|0\rangle = -\frac{1}{(2\pi)^2}\,\frac{1}{(\underline{x}-\underline{x}')^2}\,. \tag{225}$$

Skipping the y, z-parts and transforming into the Rindler coordinates introduced in Eq. (91), which are adapted to an accelerated observer, one obtains

$$\langle 0| \hat{\Phi}(\tau,\rho)\hat{\Phi}(\tau',\rho')\,|0\rangle = -\frac{1}{(2\pi)^2}\,\frac{1}{2\rho\rho'\cosh(\kappa[\tau-\tau'])-\rho^2-\rho'^2}\,. \tag{226}$$

By inspection, one observes a periodicity along imaginary τ-axis, which usually is a property of thermal states. Indeed, it can be shown (see Problems) that, if all operators (observables) $\hat{X}$ and $\hat{Y}$ (of an irreducible algebra) satisfy

$$\langle \hat{X}(\tau)\hat{Y}(\tau')\rangle_\beta = \langle \hat{Y}(\tau')\hat{X}(\tau+i\beta)\rangle_\beta \tag{227}$$

for some β, then $\langle\dots\rangle_\beta$ denotes a thermal state with the inverse temperature β. The above equality is called the Kubo-Martin-Schwinger (KMS) condition. Comparison with Eq. (226) yields Unruh temperature

$$T_{\text{Unruh}} = \frac{1}{\beta} = \frac{\kappa}{2\pi}\,, \tag{228}$$

i.e., the Minkowski vacuum that corresponds to zero temperature (no particles) for all inertial observers, exhibits thermal properties for an accelerated Rindler observer.
A more direct way to support the above statement is the explicite calculation of the number of Rindler particles in the Minkowski vacuum. This can be achieved with the Bogoliubov β-coefficients derived in Eq. (97); with the identity [...]

$$\Gamma(z)\Gamma(-z) = -\frac{\pi}{z\sin(\pi z)}\,, \tag{229}$$

we obtain

$$|\beta_{\omega,\zeta;\omega',\zeta'}|^2 = \frac{1}{2\pi\kappa\omega}\,\frac{\delta_{\zeta,\zeta'}}{\exp\{2\pi\omega'/\kappa\}-1}\,. \tag{230}$$

The second factor exactly reproduces a Bose-Einstein distribution with the Unruh temperature in Eq. (228) indicating a thermal spectrum of Rindler particles. However, in view of the first factor the remaining integration over the initial frequencies ω diverges –

[17]The specific structure of this distribution at the light cone $(\underline{x}-\underline{x}')^2 = 0$ is determined by the Wightman axioms (e.g., spectral property) and necessitates additional examinations – but does not change the conclusions.

but this is just an artefact of the use of plane waves with the divergence corresponding to the unbound volume[18] – for wave-packets one naturally obtains finite results.
As another elegant way we may define modified Bogoliubov coefficients via

$$\breve{\alpha}_{\zeta,\zeta'}(\omega,\omega') = \sqrt{\omega\omega'}\,\alpha_{\omega,\zeta;\omega',\zeta'} \quad ; \quad \breve{\beta}_{\zeta,\zeta'}(\omega,\omega') = \sqrt{\omega\omega'}\,\beta_{\omega,\zeta;\omega',\zeta'}\,, \tag{231}$$

which can be analytically continued into the complex ω-plane, where, according to Eq. (84), the relation

$$\breve{\alpha}_{\zeta,\zeta'}(\omega,\omega') = \breve{\beta}_{\zeta,\zeta'}(-\omega,\omega') \tag{232}$$

holds. Recalling Eq. (97) and taking into account the branch cut of the logarithm in the complex plane we obtain

$$\breve{\alpha}_{\zeta,\zeta'}(\omega,\omega') = \exp\left\{\frac{\pi\omega'}{\kappa}\right\}\breve{\beta}_{\zeta,\zeta'}(\omega,\omega')\,, \tag{233}$$

and thus $\beta_{\omega,\zeta;\omega',\zeta'} = \exp\{-\pi\omega'/\kappa\}\alpha_{\omega,\zeta;\omega',\zeta'}$. Insertion into the completeness relation (86) then yields

$$N^{\rm R}_{\omega',\zeta'} = \sum\!\!\!\!\!\!\int_{\omega,\zeta} |\beta_{\omega,\zeta;\omega',\zeta'}|^2 = \frac{\delta(\omega',\omega')}{\exp\{2\pi\omega'/\kappa\}-1}\,. \tag{234}$$

Again one can read off the infinite volume divergence $\delta(\omega',\omega')$ caused by the use of plane waves instead of wave-packets.
In summary, although all inertial observers in a flat space-time share the same Minkowski vacuum, this is no longer true for non-inertial, i.e., accelerated observers. Instead the Minkowski vacuum displays thermal properties for uniformly accelerated observers with the effective Unruh temperature being related to the associated surface gravity of their particle horizon via Eq. (228) – a phenomenon which is usually called the Unruh[19] effect.
One might demur that considering the expectation value of the number operator only is not enough to infer real thermal behaviour and that the derived Bose-Einstein distribution could be a mere accident. This objection is apparently supported by the observation that the state of the quantum field is the Minkowski vacuum $|0\rangle$ – i.e., a pure state instead

[18] The IR singularity $\omega \downarrow 0$ corresponds to spatial infinity $\rho \uparrow \infty$ and the UV divergence $\omega \uparrow \infty$ to the horizon $\rho \downarrow 0$. Whereas the former can be avoided by enclosing everything by a finite box the latter is genuine [...].

[19] In German, the vocable "Unruh" describes the balance-wheel of a clock – which nicely relates to both, the non-inertial motion and the concept of time, which are important for this effect. (The corresponding adjective "unruhig" means restless.) Another association could comprise the name of one of the authors.

of a mixed state (as described by a density matrix $\hat{\varrho}$) – which cannot correspond to real thermality. However, as it will become evident in the next Sections, the thermal behaviour is indeed a much more fundamental property – the accelerated Rindler observer is not able to distinguish the Minkowski vacuum from a density matrix which he/she would assign to a thermal state. One hint in this direction might be the thermal behaviour of the Wightman function when expressed in terms of the Rindler coordinates together with the fact that two-point function (for a free field) contains in view of the Wick theorem and the Wightman axioms all information about the theory.

4.7 Simple example: coupled harmonic oscillators

In order to illustrate how a pure state might exhibit thermal features we consider again a very simple example – in this case two coupled harmonic oscillators. For conceptual clarity the coupling is assumed to be switched off initially $t < 0$ and accordingly the system is described by the undisturbed static Hamiltonian

$$\hat{H}_0 = \frac{1}{2}\left(\hat{p}_1^2 + \Omega_1^2\,\hat{q}_1^2\right) + \frac{1}{2}\left(\hat{p}_2^2 + \Omega_2^2\,\hat{q}_2^2\right) \,. \tag{235}$$

Furthermore we assume the quantum system to be initially in its ground state $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$.
At $t = 0$ we switch on a small ($\epsilon \ll 1$) interaction between these two oscillators as described by the perturbation Hamiltonian

$$\hat{H}_I(t) = \epsilon\,\Omega_1\,\Omega_2\,\hat{q}_1(t)\,\hat{q}_2(t)\sin\left([\Omega_1 + \Omega_2]t\right) \,. \tag{236}$$

The time-dependence of the operators $\hat{q}_1(t)$ and $\hat{q}_2(t)$ in the above equation is generated by the undisturbed Hamiltonian $\hat{H}_0$ in Eq. (235) meaning that we work in the interaction picture.
For example, if $\hat{q}_1(t)$ and $\hat{q}_2(t)$ describe two modes of the electromagnetic field within a cavity then such a perturbation could be caused by the oscillation of one wall of cavity or the dielectric permittivity, etc. In order to simplify the perturbation Hamiltonian $\hat{H}_I(t)$ we apply the rotating wave approximation (RWA) by keeping only the terms that are in resonance[20]

$$\hat{H}_I \overset{\text{RWA}}{\approx} \frac{i\epsilon}{4}\sqrt{\Omega_1\Omega_2}\left(\hat{a}_1^\dagger\hat{a}_2^\dagger - \hat{a}_1\hat{a}_2\right) \,. \tag{237}$$

[20]I.e., where the oscillations of the operators $\hat{q}_1(t)$ and $\hat{q}_2(t)$ compensate the external time-dependence $\sin([\Omega_1 + \Omega_2]t)$. Only those terms yield significant contributions after many periods $[\Omega_1 + \Omega_2]t \gg 1$; the other ones basically average out.

Consequently, the (approximated) time-evolution operator for the state vector generates a two-mode squeezed state

$$|\xi\rangle = \hat{S}_{12}(\xi)\,|0\rangle = \exp\left\{\xi\left(\hat{a}_1^\dagger \hat{a}_2^\dagger - \hat{a}_1 \hat{a}_2\right)\right\}|0\rangle\ , \tag{238}$$

with the squeezing parameter $\xi = \epsilon\, t\, \sqrt{\Omega_1\Omega_2}/4 > 0$.
For $\xi \neq 0$ this state is entangled – i.e., although the initial ground state factorises $|0\rangle = |0\rangle_1 \otimes |0\rangle_2$ the two-mode squeezed state does not admit a product representation $|\xi\rangle \neq |\alpha\rangle_1 \otimes |\beta\rangle_2$ and contains non-vanishing correlations.
The above representation for $|\xi\rangle$ involves an exponential function of two non-commuting operators. Recollecting the Baker-Campbell-Hausdorff formula

$$\begin{aligned} e^{\hat{X}}\, e^{\hat{Y}} &= \exp\left\{\hat{X} + \hat{Y} + \frac{1}{2}[\hat{X},\hat{Y}] + \frac{1}{12}\left[\hat{X} - \hat{Y}, [\hat{X},\hat{Y}]\right] + \right. \\ &\quad \left. + \frac{1}{24}\left([\hat{X}^2,\hat{Y}^2] + 2[\hat{Y},\hat{X}\hat{Y}\hat{X}]\right) + \ldots\right\} \\ &= \exp\left\{\hat{X} + \hat{Y} + \frac{1}{2}[\hat{X},\hat{Y}] + \frac{1}{12}\left[\hat{X} - \hat{Y}, [\hat{X},\hat{Y}]\right] - \right. \\ &\quad \left. - \frac{1}{24}\left[\hat{Y},\left[\hat{X},\left[\hat{X},\hat{Y}\right]\right]\right] + \ldots\right\}, \end{aligned} \tag{239}$$

the representation in Eq. (238) does not appear to admit a trivial simplification. Nevertheless, with the aid of another useful identity

$$e^{\hat{X}}\,\hat{Y}\,e^{-\hat{X}} = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\hat{X},\hat{Y}\right]_{(n)}\ , \tag{240}$$

where the multi-commutators are defined via $[\hat{X},\hat{Y}]_{(n+1)} = [\hat{X},[\hat{X},\hat{Y}]_{(n)}]$ and $[\hat{X},\hat{Y}]_{(0)} = \hat{Y}$, we may derive (two-mode) squeezed annihilation operators via

$$\begin{aligned} \hat{a}_1^\xi &= \hat{S}_{12}(\xi)\,\hat{a}_1\,\hat{S}_{12}^\dagger(\xi) = \hat{a}_1\cosh\xi - \hat{a}_2^\dagger\sinh\xi \\ \hat{a}_2^\xi &= \hat{S}_{12}(\xi)\,\hat{a}_2\,\hat{S}_{12}^\dagger(\xi) = \hat{a}_2\cosh\xi - \hat{a}_1^\dagger\sinh\xi \end{aligned} \tag{241}$$

which have the property $\hat{a}_1^\xi\,|\xi\rangle = \hat{a}_2^\xi\,|\xi\rangle = 0$.
As the next step we introduce an *a priori* unknown function $f : \mathbb{C}^2 \to \mathbb{C}$ which uniquely determines $|\xi\rangle$ via

$$|\xi\rangle = f\left(\hat{a}_1^\dagger, \hat{a}_2^\dagger\right)|0\rangle\ . \tag{242}$$

In view of the commutation relations $[\hat{a}_1^\dagger, \hat{a}_1] = [\hat{a}_2^\dagger, \hat{a}_2] = 1$ and $[\hat{a}_1^{(\dagger)}, \hat{a}_2^{(\dagger)}] = 0$, the conditions $\hat{a}_1^\xi\,|\xi\rangle = \hat{a}_2^\xi\,|\xi\rangle = 0$ with the (two-mode) squeezed operators given by Eq. (241)

are formally equivalent to the following differential equations

$$\left(\frac{\partial}{\partial\hat{a}_1^\dagger} - \hat{a}_2^\dagger \tanh\xi\right) f\left(\hat{a}_1^\dagger, \hat{a}_2^\dagger\right)|0\rangle = 0\,,$$
$$\left(\frac{\partial}{\partial\hat{a}_2^\dagger} - \hat{a}_1^\dagger \tanh\xi\right) f\left(\hat{a}_1^\dagger, \hat{a}_2^\dagger\right)|0\rangle = 0\,. \tag{243}$$

Together with the normalisation condition $\langle\xi|\xi\rangle = 1$ the solution of these differential equations is given by

$$|\xi\rangle = \frac{1}{\cosh\xi}\exp\left\{\hat{a}_1^\dagger\hat{a}_2^\dagger\tanh\xi\right\}|0\rangle\ , \tag{244}$$

and hence we finally arrive at

$$|\xi\rangle = \frac{1}{\cosh\xi}\sum_{n=0}^{\infty}(\tanh\xi)^n\ |n\rangle_1\otimes|n\rangle_2\ . \tag{245}$$

This representation is very instructive and demonstrates directly the strong correlations of this (pure) state – the (measured) number n of particles in first oscillator $\{1\}$ always equals that in the second one $\{2\}$.
But let us confine our attention to only one oscillator, say the first $\{1\}$, and assume that we are not interested in any observables associated with the other one – the second $\{2\}$. For example, the two systems $\{1\}$ and $\{2\}$ could be separated by a large (spatial) distance and we are able to perform measurements on oscillator $\{1\}$ only. In this case we may as well average over the degrees of freedom of the other system $\{2\}$, which does not change any of our observations. Formally, this procedure introduces an observation level $\mathcal{G}$ containing all interesting or accessible observables only, for which the state can be described by the effective density matrix [Fick & Sauermann, 1983]

$$\hat\rho_1 = \mathrm{Tr}_2\left\{\hat\rho\right\} = \mathrm{Tr}_2\left\{|\xi\rangle\langle\xi|\right\} = \frac{1}{\cosh^2\xi}\sum_{n=0}^{\infty}(\tanh\xi)^{2n}\ |n\rangle\langle n|\ , \tag{246}$$

where the trace Tr_2 averages over all degrees of freedom associated with the second oscillator (i.e., the reservoir)

$$\mathrm{Tr}_2\left\{\hat{X}_{12}\right\} = \sum_{n=0}^{\infty}\langle n|\,\hat{X}_{12}\,|n\rangle_2\ . \tag{247}$$

If we identify the temperature via

$$\tanh^2\xi = e^{-\beta\Omega_1}\ \rightsquigarrow\ T = \frac{1}{\beta} = \frac{\Omega_1}{2\ln(\coth\xi)}\,, \tag{248}$$

the effective density matrix exactly corresponds to a canonical ensemble indicating a thermal equilibrium state of the first oscillator

$$\hat{\rho}_1 = \frac{\exp\left\{-\beta \hat{H}_0^{\{1\}}\right\}}{Z_1} . \tag{249}$$

This – perhaps surprising – result is the main content of the so-called thermo-field mechanism [...], i.e., a pure (entangled) state can effectively behave as a mixed thermal state after averaging (tracing out) over degrees of freedom. The non-vanishing entropy $S_1 = -\mathrm{Tr}_1\left\{\hat{\rho}_1 \ln \hat{\rho}_1\right\}$ is called the entanglement entropy[21] and measures the correlations that are lost by the averaging process. If S_1 vanishes then the initial state (which is still assumed to be pure) is not entangled and thus can be factorised $|\Psi\rangle = |\alpha\rangle_1 \otimes |\beta\rangle_2$ (no correlations).

With observations on one system only ($\mathcal{G} = \{1\}$) one can never find out the difference between this effective thermality and real thermality. For instance the expectation value of the number operator

$$\mathrm{Tr}_1\left\{\hat{\rho}_1\,\hat{n}_1\right\} = \langle\xi|\,\hat{n}_1\,|\xi\rangle = \sinh^2\xi = \frac{1}{\exp\left\{\beta\Omega_1\right\} - 1} , \tag{250}$$

exactly matches the thermal result. Measurements on both systems, however, easily reveal the non-thermal nature of the state: remember, for example, the perfect correlation $n_1 = n_2$ noticed in Eq. (245) – whereas in a real thermal state, these number would be completely uncorrelated $\langle\hat{n}_1\hat{n}_2\rangle = \langle\hat{n}_1\rangle\langle\hat{n}_2\rangle$.

As we have observed, the above equation can be derived equally well with the Bogoliubov transformation and thus the effective temperature T depends on the squeezing parameter ξ, i.e., on the Bogoliubov coefficients, via Eq. (248). If our observation level includes more than one oscillators with different frequencies then the possibility of introducing *one* effective temperature (i.e., for all oscillators the same T) evidently requires strong relation between the Bogoliubov coefficients, cf. Eq. (248).

As we shall see later, for the Unruh and (late-time part of the) the Hawking effect this requirement is indeed satisfied, since the relevant Bogoliubov coefficients fulfil $|\beta_{\omega\omega'}| = \exp\{-\pi\omega'/\kappa\}|\alpha_{\omega\omega'}|$. In these situations the effective statistical operator can be obtained from the (initial) pure state by averaging over the degrees of freedom beyond the horizon – which are inaccessible to the observer under consideration.

[21] This quantity also plays an important rôle in quantum information theory.

4.8 Gaussian states and squeezing

In fact, the correspondence discussed above is not restricted to two coupled harmonic oscillators but applies to all Gaussian states. A Gaussian state is a quantum state with a Gaussian wave function

$$\psi(\boldsymbol{x}) = \mathcal{N} \exp\left\{-\frac{1}{2}\,\boldsymbol{x}\cdot\boldsymbol{M}\cdot\boldsymbol{x}\right\} = \mathcal{N} \exp\left\{-\frac{1}{2}\,x_I M_{IJ} x_J\right\}, \tag{251}$$

where $\mathcal{N}$ is the normalisation factor ensuring $\langle\psi|\psi\rangle = 1$, $\boldsymbol{x} = \{x_J\}$ are the continuous (position) coordinates, and $\boldsymbol{M} = \{M_{IJ}\}$ is a symmetric matrix. Since $\psi(\boldsymbol{x})$ has to be square integrable $\langle\psi|\psi\rangle = 1$, the eigenvalues of the real part of the matrix $\boldsymbol{M}$ must all be positive. The imaginary part of $\boldsymbol{M}$ can be arbitrary (as long as it is symmetric). Taking $\boldsymbol{x}$-derivatives of $\psi(\boldsymbol{x})$, we find

$$\frac{\partial}{\partial \boldsymbol{x}}\,\psi(\boldsymbol{x}) = -\boldsymbol{M}\cdot\boldsymbol{x}\,\psi(\boldsymbol{x}) \;\rightsquigarrow\; (\hat{\boldsymbol{p}} - i\boldsymbol{M}\cdot\hat{\boldsymbol{x}})\,|\psi\rangle = 0\,, \tag{252}$$

with the canonical momentum operators $\hat{\boldsymbol{p}} = \{\hat{p}_I\}$. This suggests introducing the "pre-annihilation" operators $\hat{\boldsymbol{A}} = \hat{\boldsymbol{p}} - i\boldsymbol{M}\cdot\hat{\boldsymbol{x}}$, which obey the commutation relations

$$\left[\hat{A}_I, \hat{A}_J\right] = \left[\hat{A}_I^\dagger, \hat{A}_J^\dagger\right] = 0\,,\; \left[\hat{A}_I, \hat{A}_J^\dagger\right] = M_{IJ} + M_{IJ}^* = 2\Re(M_{IJ})\,. \tag{253}$$

Since $\Re(M_{IJ})$ is a symmetric and positive matrix, we may diagonalise it with an orthogonal (rotation) matrix $\boldsymbol{D}$ such that $\boldsymbol{D}\cdot\Re(\boldsymbol{M})\cdot\boldsymbol{D}^\dagger = \mathrm{diag}\{\lambda_I\}$. As a result, the operators

$$\hat{a}_I = \frac{D_{IJ}\hat{A}_J}{\sqrt{2\lambda_I}} \;\rightsquigarrow\; \hat{a}_I\,|\psi\rangle = 0 \tag{254}$$

satisfy the commutation relations (204) and hence can be interpreted as creation and annihilation operators for which $|\psi\rangle$ is the vacuum state. As a result, all Gaussian states (251) are vacuum states associated to some set of creation and annihilation operators.
The reverse is also true. In order to prove this, let us consider two different vacuum states $|0\rangle$ and $|0'\rangle$ where the associated sets of creation and annihilation operators are related to each other via the Bogoliubov coefficients α_{IJ} and β_{IJ}

$$\hat{a}_I\,|0\rangle = 0\,,\; \hat{a}'_I\,|0'\rangle = \sum_J \left(\alpha_{IJ}\hat{a}_J + \beta_{IJ}\hat{a}_J^\dagger\right)|0'\rangle = 0\,. \tag{255}$$

In position representation, this becomes

$$\begin{aligned} \left(\boldsymbol{\alpha}\cdot\hat{\boldsymbol{a}} + \boldsymbol{\beta}\cdot\hat{\boldsymbol{a}}^\dagger\right)|0'\rangle = (\boldsymbol{\alpha}\cdot[\hat{\boldsymbol{x}} + i\hat{\boldsymbol{p}}] + \boldsymbol{\beta}\cdot[\hat{\boldsymbol{x}} - i\hat{\boldsymbol{p}}])\,|0'\rangle &= 0 \\ \rightsquigarrow\; \left([\boldsymbol{\alpha} - \boldsymbol{\beta}]\cdot\frac{\partial}{\partial\boldsymbol{x}} + [\boldsymbol{\alpha} + \boldsymbol{\beta}]\cdot\boldsymbol{x}\right)\psi'_0(\boldsymbol{x}) &= 0\,. \end{aligned} \tag{256}$$

Obviously, the solution of this equation is a Gaussian state (251). Note that $\boldsymbol{\alpha}+\boldsymbol{\beta}$ and $\boldsymbol{\alpha}-\boldsymbol{\beta}$ are both invertible matrices $\det\{\boldsymbol{\alpha}\pm\boldsymbol{\beta}\}\neq 0$. This can be shown by assuming the existence of a non-vanishing eigenvector $\boldsymbol{n}_0$ with zero eigenvalue $\boldsymbol{n}_0^\dagger\cdot(\boldsymbol{\alpha}\pm\boldsymbol{\beta})=0$ which would imply $\boldsymbol{n}_0^\dagger\cdot\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}^\dagger\cdot\boldsymbol{n}_0=\boldsymbol{n}_0^\dagger\cdot\boldsymbol{\beta}\cdot\boldsymbol{\beta}^\dagger\cdot\boldsymbol{n}_0$. However, this contradicts the completeness relation $\boldsymbol{\alpha}\cdot\boldsymbol{\alpha}^\dagger=\mathbf{1}+\boldsymbol{\beta}\cdot\boldsymbol{\beta}^\dagger$ of the Bogoliubov coefficients in (86). In summary, all Gaussian states are states which are annihilated by a suitable complete set of annihilation operators, i.e., vacuum states (of a free quantum field theory) and *vice versa*.

Moreover, all multi-mode squeezed states are vacuum states and thus also Gaussian states (251). Starting with the most general definition of a (multi-mode) squeezed state

$$|\psi_{\boldsymbol{\xi},\boldsymbol{\chi}}\rangle=\exp\left\{\frac{1}{2}\hat{\boldsymbol{a}}^\dagger\cdot\boldsymbol{\xi}\cdot\hat{\boldsymbol{a}}^\dagger+\frac{i}{2}\hat{\boldsymbol{a}}^\dagger\cdot\boldsymbol{\chi}\cdot\hat{\boldsymbol{a}}-\text{h.c.}\right\}|0\rangle=\hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}}|0\rangle\,, \tag{257}$$

with arbitrary matrices $\boldsymbol{\xi}$ and $\boldsymbol{\chi}$, we introduce the generalised squeezing operator $\hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}}$ which is unitary. With this operator, we may transform the creation and annihilation operators via $\hat{a}_I'=\hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}}\hat{a}_I\hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}}^\dagger$. With the aid of the identity (240), we get

$$\begin{aligned}\hat{\boldsymbol{a}}' &= \hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}}\,\hat{\boldsymbol{a}}\,\hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}}^\dagger=\sum_{n=0}^\infty\frac{1}{n!}\left[\frac{1}{2}\hat{\boldsymbol{a}}^\dagger\cdot\boldsymbol{\xi}\cdot\hat{\boldsymbol{a}}^\dagger+\frac{i}{2}\hat{\boldsymbol{a}}^\dagger\cdot\boldsymbol{\chi}\cdot\hat{\boldsymbol{a}}-\text{h.c.},\hat{\boldsymbol{a}}\right]_{(n)}\\ &= \boldsymbol{\alpha}\cdot\hat{\boldsymbol{a}}+\boldsymbol{\beta}\cdot\hat{\boldsymbol{a}}^\dagger\,,\end{aligned} \tag{258}$$

where $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are infinite power series of the matrices $\boldsymbol{\xi}$ and $\boldsymbol{\chi}$, see Eq. (267) below. In some cases, it is possible to sum this infinite series arriving at a simpler expression. To this end, let us introduce the auxiliary variable τ with

$$\hat{\boldsymbol{a}}(\tau)=\hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}}(\tau)\hat{\boldsymbol{a}}\hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}}^\dagger(\tau)\,, \tag{259}$$

where

$$\hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}}(\tau)=\hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}}^\tau=\exp\left\{\frac{\tau}{2}\left(\hat{\boldsymbol{a}}^\dagger\cdot\boldsymbol{\xi}\cdot\hat{\boldsymbol{a}}^\dagger+i\hat{\boldsymbol{a}}^\dagger\cdot\boldsymbol{\chi}\cdot\hat{\boldsymbol{a}}-\text{h.c.}\right)\right\}\,. \tag{260}$$

Then we get the following differential equations for $\hat{\boldsymbol{a}}(\tau)$

$$\frac{d}{d\tau}\,\hat{\boldsymbol{a}}(\tau)=-\boldsymbol{\xi}\cdot\hat{\boldsymbol{a}}^\dagger(\tau)-i\boldsymbol{\chi}\cdot\hat{\boldsymbol{a}}(\tau)\,, \tag{261}$$

where we have assumed that $\boldsymbol{\xi}$ is symmetric $\boldsymbol{\xi}=\boldsymbol{\xi}^T$ and that $\boldsymbol{\chi}$ is self-adjoint $\boldsymbol{\chi}=\boldsymbol{\chi}^\dagger$. The second derivative then becomes

$$\frac{d^2}{d\tau^2}\,\hat{\boldsymbol{a}}(\tau)=\left(\boldsymbol{\xi}\cdot\boldsymbol{\xi}^*-\boldsymbol{\chi}^2\right)\cdot\hat{\boldsymbol{a}}(\tau)+i\left(\boldsymbol{\chi}\cdot\boldsymbol{\xi}-\boldsymbol{\xi}\cdot\boldsymbol{\chi}^*\right)\cdot\hat{\boldsymbol{a}}^\dagger(\tau)\,. \tag{262}$$

If the second term vanishes, i.e., if $\boldsymbol{\chi}\cdot\boldsymbol{\xi}=\boldsymbol{\xi}\cdot\boldsymbol{\chi}^*$ holds, this equation can be solved explicitly in terms of the matrix functions $\cosh(\tau\sqrt{\boldsymbol{\xi}\cdot\boldsymbol{\xi}^*-\boldsymbol{\chi}^2})$ and $\sinh(\tau\sqrt{\boldsymbol{\xi}\cdot\boldsymbol{\xi}^*-\boldsymbol{\chi}^2})$. Note that $\boldsymbol{\xi}\cdot\boldsymbol{\xi}^*-\boldsymbol{\chi}^2=\boldsymbol{\xi}\cdot\boldsymbol{\xi}^\dagger-\boldsymbol{\chi}\cdot\boldsymbol{\chi}^\dagger$ is self-adjoint and hence can be diagonalised.

Obviously, the operators $\hat{\boldsymbol{a}}' = \hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}} \hat{\boldsymbol{a}} \hat{U}^\dagger_{\boldsymbol{\xi},\boldsymbol{\chi}}$ annihilate the squeezed state (257) demonstrating that it is a vacuum state. Again, the reverse statement – i.e., that every vacuum state $|0'\rangle$ is a squeezed state (257) with respect to all other vacuum states – can be shown as well. To see this, it is useful to employ the polar decomposition of the matrices $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$

$$\boldsymbol{\alpha} = |\boldsymbol{\alpha}| \cdot \boldsymbol{u}_\alpha = \sqrt{\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}^\dagger} \cdot \boldsymbol{u}_\alpha \,, \quad \boldsymbol{\beta} = |\boldsymbol{\beta}| \cdot \boldsymbol{u}_\beta = \sqrt{\boldsymbol{\beta} \cdot \boldsymbol{\beta}^\dagger} \cdot \boldsymbol{u}_\beta \,, \tag{263}$$

into self-adjoint and non-negative matrices $|\boldsymbol{\alpha}|$ and $|\boldsymbol{\beta}|$ as well as unitary matrices $\boldsymbol{u}_\alpha$ and $\boldsymbol{u}_\beta$, respectively. Inserting this decomposition into the vacuum definition

$$\left(\boldsymbol{\alpha} \cdot \hat{\boldsymbol{a}} + \boldsymbol{\beta} \cdot \hat{\boldsymbol{a}}^\dagger\right) |0'\rangle = \left(|\boldsymbol{\alpha}| \cdot \boldsymbol{u}_\alpha \cdot \hat{\boldsymbol{a}} + |\boldsymbol{\beta}| \cdot \boldsymbol{u}_\beta \cdot \hat{\boldsymbol{a}}^\dagger\right) |0'\rangle = 0 \,, \tag{264}$$

we may absorb the unitary matrix $\boldsymbol{u}_\alpha$ by a redefinition of the annihilation operators via $\hat{\boldsymbol{a}} \to \boldsymbol{u}_\alpha \cdot \hat{\boldsymbol{a}}$ which leaves the vacuum state $|0'\rangle$ invariant. This transformation corresponds to the $\boldsymbol{\chi}$-part of the operator $\hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}}$ in (257)

$$\begin{aligned} \hat{U}_{\boldsymbol{\chi}} \, \hat{\boldsymbol{a}} \, \hat{U}^\dagger_{\boldsymbol{\chi}} &= \exp\left\{i\hat{\boldsymbol{a}}^\dagger \cdot \boldsymbol{\chi} \cdot \hat{\boldsymbol{a}}\right\} \hat{\boldsymbol{a}} \exp\left\{-i\hat{\boldsymbol{a}}^\dagger \cdot \boldsymbol{\chi} \cdot \hat{\boldsymbol{a}}\right\} = \sum_{n=0}^\infty \frac{1}{n!} \left[i\hat{\boldsymbol{a}}^\dagger \cdot \boldsymbol{\chi} \cdot \hat{\boldsymbol{a}}, \hat{\boldsymbol{a}}\right]_{(n)} \\ &= \exp\{-i\boldsymbol{\chi}\} \cdot \hat{\boldsymbol{a}} = \boldsymbol{u}_\alpha \cdot \hat{\boldsymbol{a}} \,. \end{aligned} \tag{265}$$

Here we have used the identity (240) and the fact that every unitary matrix $\boldsymbol{u}_\alpha$ can be written as $\boldsymbol{u}_\alpha = \exp\{-i\boldsymbol{\chi}\}$ with some self-adjoint matrix $\boldsymbol{\chi}$.
Furthermore, the completeness relation $\boldsymbol{\alpha} \cdot \boldsymbol{\alpha}^\dagger = \mathbf{1} + \boldsymbol{\beta} \cdot \boldsymbol{\beta}^\dagger$ of the Bogoliubov coefficients in (86) shows that $|\boldsymbol{\alpha}|$ and $|\boldsymbol{\beta}|$ can be diagonalised simultaneously and satisfy $|\boldsymbol{\alpha}|^2 = \mathbf{1} + |\boldsymbol{\beta}|^2$. Consequently, we may write them as $|\boldsymbol{\alpha}| = \cosh(|\boldsymbol{\xi}|)$ and $|\boldsymbol{\beta}| = \sinh(|\boldsymbol{\xi}|)$ with some self-adjoint and non-negative matrix $\boldsymbol{\xi}$. Altogether, this gives

$$\left(\cosh(|\boldsymbol{\xi}|) \cdot \hat{\boldsymbol{a}} + \sinh(|\boldsymbol{\xi}|) \cdot \boldsymbol{u}_\xi \cdot \hat{\boldsymbol{a}}^\dagger\right) |0'\rangle = \left(\hat{\boldsymbol{a}} + \tanh(|\boldsymbol{\xi}|) \cdot \boldsymbol{u}_\xi \cdot \hat{\boldsymbol{a}}^\dagger\right) |0'\rangle = 0 \,, \tag{266}$$

with some unitary matrix $\boldsymbol{u}_\xi$. Now it remains to be shown that this is indeed a squeezed state. To this end, we again use the identity (240)

$$\begin{aligned} \hat{U}_{\boldsymbol{\xi}} \, \hat{\boldsymbol{a}} \, \hat{U}^\dagger_{\boldsymbol{\xi}} &= \sum_{n=0}^\infty \frac{1}{n!} \left[\frac{1}{2}\hat{\boldsymbol{a}}^\dagger \cdot \boldsymbol{\xi} \cdot \hat{\boldsymbol{a}}^\dagger - \frac{1}{2}\hat{\boldsymbol{a}} \cdot \boldsymbol{\xi}^\dagger \cdot \hat{\boldsymbol{a}} \,,\, \hat{\boldsymbol{a}}\right]_{(n)} \\ &= \hat{\boldsymbol{a}} + \boldsymbol{\xi} \cdot \hat{\boldsymbol{a}}^\dagger + \frac{1}{2} \boldsymbol{\xi} \cdot \boldsymbol{\xi}^\dagger \cdot \hat{\boldsymbol{a}} + \frac{1}{3!} \boldsymbol{\xi} \cdot \boldsymbol{\xi}^\dagger \cdot \boldsymbol{\xi} \cdot \hat{\boldsymbol{a}}^\dagger + \frac{1}{4!} \left(\boldsymbol{\xi} \cdot \boldsymbol{\xi}^\dagger\right)^2 \cdot \hat{\boldsymbol{a}} + \dots \\ &= \cosh(|\boldsymbol{\xi}|) \cdot \hat{\boldsymbol{a}} + \sinh(|\boldsymbol{\xi}|) \cdot \boldsymbol{u}_\xi \cdot \hat{\boldsymbol{a}}^\dagger \,, \end{aligned} \tag{267}$$

and find that $\hat{U}_{\boldsymbol{\xi}} \, \hat{\boldsymbol{a}} \, \hat{U}^\dagger_{\boldsymbol{\xi}}$ annihilates the state $|0'\rangle$. Therefore, the state $\hat{U}^\dagger_{\boldsymbol{\xi}} \, |0'\rangle$ is annihilated by $\hat{\boldsymbol{a}}$, i.e., it is the vacuum state $|0\rangle$, which in turn implies $|0'\rangle = \hat{U}_{\boldsymbol{\xi}} \, |0\rangle$. Note that the matrix $\boldsymbol{\xi}$ in the above equation is symmetric $\boldsymbol{\xi} = \boldsymbol{\xi}^T$. This is ensured by the second consistency relation of the Bogoliubov coefficients $\boldsymbol{\alpha} \cdot \boldsymbol{\beta}^{\mathrm{T}} - \boldsymbol{\beta} \cdot \boldsymbol{\alpha}^{\mathrm{T}} = 0$ which implies $\boldsymbol{u}_\xi^T \cdot |\boldsymbol{\xi}| \cdot \boldsymbol{u}_\xi = |\boldsymbol{\xi}|^T$ and $\boldsymbol{u}_\xi^T \cdot \tanh|\boldsymbol{\xi}| \cdot \boldsymbol{u}_\xi = \tanh|\boldsymbol{\xi}|^T$ etc.

The above proof does not only apply to the states, but to the full transformation. With exactly the same steps, one can show that a generalised squeezing operator $\hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}}$ in (257) with arbitrary matrices $\boldsymbol{\xi}$ and $\boldsymbol{\chi}$ always generates a Bogoliubov transformation $\hat{\boldsymbol{a}} \to \boldsymbol{\alpha} \cdot \hat{\boldsymbol{a}} + \boldsymbol{\beta} \cdot \hat{\boldsymbol{a}}^\dagger$ with some Bogoliubov coefficients $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$. Conversely, every Bogoliubov transformation can be generated by a pure rotation $\hat{U}_{\boldsymbol{\chi}} = \exp\left\{i\hat{\boldsymbol{a}}^\dagger \cdot \boldsymbol{\chi} \cdot \hat{\boldsymbol{a}}\right\}$ with some self-adjoint matrix $\boldsymbol{\chi}$ followed by a pure squeezing transformation $\hat{U}_{\boldsymbol{\xi}} = \exp\{\hat{\boldsymbol{a}}^\dagger \cdot \boldsymbol{\xi} \cdot \hat{\boldsymbol{a}}^\dagger/2 - \text{h.c.}\}$ with a symmetric matrix $\boldsymbol{\xi}$. In view of the Baker-Campbell-Hausdorff relation (239), the product $\hat{U}_{\boldsymbol{\xi}}\hat{U}_{\boldsymbol{\chi}}$ can be written as a generalised squeezing operator $\hat{U}_{\boldsymbol{\xi}',\boldsymbol{\chi}'}$ in (257), but with different matrices $\boldsymbol{\xi}'$ and $\boldsymbol{\chi}'$ in general. As a by-product, the above proof also shows that every $\hat{U}_{\boldsymbol{\xi}',\boldsymbol{\chi}'}$ can be factorised $\hat{U}_{\boldsymbol{\xi}',\boldsymbol{\chi}'} = \hat{U}_{\boldsymbol{\xi}}\hat{U}_{\boldsymbol{\chi}}$, see also [Ma & Rhodes, 1990].

Sometimes it is useful to employ yet another representation. Since every state $|\psi\rangle$ can be created by acting an appropriate function $f(\hat{\boldsymbol{a}}^\dagger)$ on the vacuum $|0\rangle$, we may express another vacuum state $|0'\rangle$ in this way

$$|0'\rangle = f(\hat{\boldsymbol{a}}^\dagger)\,|0\rangle\;. \tag{268}$$

The function f can be obtained from Eq. (266) which yields

$$\left(\hat{\boldsymbol{a}} + \tanh(|\boldsymbol{\xi}|) \cdot \boldsymbol{u}_\xi \cdot \hat{\boldsymbol{a}}^\dagger\right)|0'\rangle = \left(\frac{\partial}{\partial \hat{\boldsymbol{a}}^\dagger} + \tanh(|\boldsymbol{\xi}|) \cdot \boldsymbol{u}_\xi \cdot \hat{\boldsymbol{a}}^\dagger\right) f(\hat{\boldsymbol{a}}^\dagger)\,|0\rangle = 0\,. \tag{269}$$

Since all the creation operators $\hat{a}_J^\dagger$ commute, this equation can be solved in the same way as a differential equation with c-numbers and has the solution

$$f(\hat{\boldsymbol{a}}^\dagger) = \mathcal{N} \exp\left\{-\frac{1}{2}\,\hat{\boldsymbol{a}}^\dagger \cdot \tanh(|\boldsymbol{\xi}|) \cdot \boldsymbol{u}_\xi \cdot \hat{\boldsymbol{a}}^\dagger\right\}\;. \tag{270}$$

Consequently, every vacuum state $|0'\rangle$ can be created out of any other vacuum via

$$|0'\rangle = \mathcal{N} \exp\left\{\frac{1}{2}\,\hat{\boldsymbol{a}}^\dagger \cdot \Xi \cdot \hat{\boldsymbol{a}}^\dagger\right\}|0\rangle = \mathcal{N} \exp\left\{\frac{1}{2}\sum_{IJ} \hat{a}_I^\dagger \Xi_{IJ} \hat{a}_J^\dagger\right\}|0\rangle\;, \tag{271}$$

with an appropriate matrix $\Xi = \tanh(|\boldsymbol{\xi}|) \cdot \boldsymbol{u}_\xi$.

Note that, for linear field equations, the time evolution operator $\hat{U}(t)$ is always a (multi-mode) squeezing operator $\hat{U}_{\boldsymbol{\xi},\boldsymbol{\chi}}$. As a result, all fundamental quantum effects of linear fields – such as Hawking radiation, the Unruh effect, cosmological particle creation, but also the Schwinger mechanism and others – can be understood as generalised squeezing.

We now have enough to describe and study linear amplifiers. We will look at two types – phase sensitive amplifiers and general (i.e., phase insensitive) amplifiers. For the first, the phase sensitive amplifier, we need to only look at a single oscillator with annihilation operator $\hat{a}$. We have the general squeezing operator $\hat{U}_{\xi,\chi}$ such that

$$\hat{U}^\dagger_{\xi,\chi}\hat{a}\hat{U}_{\xi,\chi} = \cosh(\xi)e^{i\chi}\hat{a} + \sinh(\xi)e^{-i\chi}\hat{a}^\dagger \,. \tag{272}$$

We also have the displacement operator, $\hat{D}_\alpha$, such that

$$\hat{D}^\dagger_\alpha\hat{a}\hat{D}_\alpha = \hat{a} + \alpha \,. \tag{273}$$

Now consider the coherent state

$$|\alpha\rangle = \hat{D}_\alpha |0\rangle \,, \tag{274}$$

which we use to describe the (classical) signal to be amplified. We now put this state through some process – an amplifier, that takes it to the new state

$$|\psi\rangle = \hat{U}_{\xi,\chi} |\alpha\rangle \,. \tag{275}$$

The expectation value of $\hat{a}$ in this state is

$$\begin{aligned} \langle\psi| \hat{a} |\psi\rangle &= \langle\alpha| \hat{U}^\dagger_{\xi,\chi}\hat{a}\hat{U}_{\xi,\chi} |\alpha\rangle = \langle\alpha| \cosh(\xi)e^{i\chi}\hat{a} + \sinh(\xi)e^{-i\chi}\hat{a}^\dagger |\alpha\rangle \\ &= \cosh(\xi)e^{i\chi}\alpha + \sinh(\xi)e^{i\chi}\alpha^* \,. \end{aligned} \tag{276}$$

The result depends on the phase of α. For simplicity, we assume $\xi > 0$ and $\chi = 0$. If α is real, the result is αe^{ξ}. I.e., it is much larger than the incoming displacement α. If α is imaginary, we get $\alpha e^{-\xi}$, which is much smaller than the original. This is a phase sensitive detector – it amplifies or de-amplifies the signal depending on the phase of the signal. Since we cannot measure $\hat{a}$, we can look at its real and imaginary parts,

$$\hat{x} = \frac{1}{\sqrt{2}}(\hat{a}^\dagger + \hat{a}) \,, \; \hat{p} = \frac{i}{\sqrt{2}}(\hat{a}^\dagger - \hat{a}) \,. \tag{277}$$

The expectation value of $\hat{x}$ and $\hat{p}$ and their uncertainty before the amplification is

$$\langle\alpha| \hat{x} |\alpha\rangle = \sqrt{2}\Re(\alpha) \,, \; \langle\alpha| \hat{p} |\alpha\rangle = \sqrt{2}\Im(\alpha) \,, \; \Delta x^2 = \langle\alpha| \hat{x}^2 |\alpha\rangle - \langle\alpha| \hat{x} |\alpha\rangle^2 = \frac{1}{2} \,, \tag{278}$$

and similarly $\Delta p^2 = 1/2$, while afterwards it is

$$\langle\psi| \hat{x} |\psi\rangle = \sqrt{2}e^{\xi}\Re(\alpha) \,, \; \langle\psi| \hat{p} |\psi\rangle = \sqrt{2}e^{-\xi}\Im(\alpha) \,, \; \Delta x^2 = \frac{e^{2\xi}}{2} \,, \; \Delta p^2 = \frac{e^{-2\xi}}{2} \,. \tag{279}$$

The signal to noise ratios of both the amplified $\langle x\rangle/\Delta x$ and de-amplified $\langle p\rangle/\Delta p$ directions remains the same after amplification for such a phase sensitive detector. Before and after amplification, the state saturates the Heisenberg uncertainty relation $\Delta x\Delta p = 1/2$.

The alternative is to use the two mode squeezing operation which will be a model for the phase insensitive detector. In this case we have two oscillators, whose annihilation operators are designated by $\hat{a}$ and $\hat{b}$. We will assume that the $\hat{a}$ oscillator carries the signal, and that $\hat{b}$ corresponds to an auxiliary idler oscillator. We now have the two mode squeezing operator $\hat{U}_{\boldsymbol{\xi}}$ with the matrix

$$\boldsymbol{\xi} = \xi\sigma_x = \xi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{280}$$

which generates the transformation

$$\hat{U}_{\boldsymbol{\xi}}^\dagger \hat{a} \hat{U}_{\boldsymbol{\xi}} = \cosh(\xi)\hat{a} - \sinh(\xi)\hat{b}^\dagger , \quad \hat{U}_{\boldsymbol{\xi}}^\dagger \hat{b} \hat{U}_{\boldsymbol{\xi}} = \cosh(\xi)\hat{b} - \sinh(\xi)\hat{a}^\dagger \tag{281}$$

There is a more general squeezing operator (257) with arbitrary matrices $\boldsymbol{\xi}$ and $\boldsymbol{\chi}$ but it adds nothing essential to this model. We again start with the state where the $\hat{a}$ oscillator is in a coherent state with displacement α while the $\hat{b}$ oscillator is in its ground state

$$|\alpha\rangle = |\alpha\rangle_a \otimes |0\rangle_b = \hat{D}_\alpha |0\rangle_a \otimes |0\rangle_b \ . \tag{282}$$

Now, after applying the squeezing $|\psi\rangle = \hat{U}_{\boldsymbol{\xi}} |\alpha\rangle$, we have

$$\begin{aligned} \langle\psi| \hat{a} |\psi\rangle &= \langle\alpha| \cosh(\xi)\hat{a} - \sinh(\xi)\hat{b}^\dagger |\alpha\rangle = \cosh(\xi)\alpha \\ \langle\psi| \hat{a}^\dagger |\psi\rangle &= \langle\alpha| \cosh(\xi)\hat{a}^\dagger - \sinh(\xi)\hat{b} |\alpha\rangle = \cosh(\xi)\alpha^* \end{aligned} \tag{283}$$

We note that – in contrast to the case of single mode squeezing – the expectation values of $\hat{a}$ and $\hat{a}^\dagger$ are multiplied by the same factor $\cosh(\xi)$. Consequently, we get

$$\begin{aligned} \langle\psi| \hat{x} |\psi\rangle &= \sqrt{2}\cosh(\xi)\Re(\alpha) = \cosh(\xi) \langle\alpha| \hat{x} |\alpha\rangle \ , \\ \langle\psi| \hat{p} |\psi\rangle &= \sqrt{2}\cosh(\xi)\Im(\alpha) = \cosh(\xi) \langle\alpha| \hat{p} |\alpha\rangle \ , \end{aligned} \tag{284}$$

i.e., both phases of the signal are multiplied by the same factor $\cosh(\xi)$. Thus it is a phase insensitive amplification.

The uncertainties in $\hat{x} = (\hat{a}^\dagger + \hat{a})/\sqrt{2}$ and $\hat{p} = i(\hat{a}^\dagger - \hat{a})/\sqrt{2}$ are now equal

$$\begin{aligned} \Delta x^2 &= \langle\psi| \hat{x}^2 |\psi\rangle - (\langle\psi| \hat{x} |\psi\rangle)^2 \\ &= \frac{1}{2} \langle\alpha| [\cosh(\xi)(\hat{a} + \hat{a}^\dagger) - \sinh(\xi)(\hat{b} + \hat{b}^\dagger)]^2 |\alpha\rangle - 2[\cosh(\xi)\Re(\alpha)]^2 \\ &= \frac{\cosh^2(\xi) + \sinh^2(\xi)}{2} = \frac{\cosh(2\xi)}{2} = \Delta p^2 \ . \end{aligned} \tag{285}$$

We note that in this case, the signal to noise ratio is degraded

$$\frac{\langle\hat{x}\rangle}{\Delta x} = 2\Re(\alpha)\frac{\cosh(\xi)}{\sqrt{\cosh(2\xi)}} \ . \tag{286}$$

For large amplification $\xi \gg 1$, the noise is increased by a factor of $e^{\xi}/\sqrt{2}$ for both phases of the signal, while the signal is amplified by only a factor of $e^{\xi}/2$. A phase insensitive amplifier is always noisy (i.e., it adds noise and reduces the signal to noise ratio).

Even if the incoming signal in the $\hat{a}$-channel is the vacuum state $\alpha = 0$, the outgoing $\hat{a}$ channel is noisy. Furthermore, the outgoing $\hat{a}$-channel is in a mixed, not a pure state. As we have shown, every squeezed state is a Gaussian state, in this case in the coordinates $\hat{x}$ and $\hat{y} = (\hat{b}^{\dagger} + \hat{b})/\sqrt{2}$. When one traces out over some of the variables of a Gaussian density matrix, one gets a Gaussian in the reduced variables. This reduced density matrix can be written as

$$\hat{\rho} = \frac{1}{Z} \exp\left\{-[A\hat{x}^2 + B\hat{p}^2 + C(\hat{x}\hat{p} + \hat{p}\hat{x})]\right\} , \tag{287}$$

where Z ensures $\mathrm{Tr}\{\hat{\rho}\} = 1$

$$Z = \mathrm{Tr}\left\{e^{-[A\hat{x}^2 + B\hat{p}^2 + C(\hat{x}\hat{p} + \hat{p}\hat{x})]}\right\} . \tag{288}$$

This trace is most easily evaluated by regarding the operator in the exponent as if it were a Hamiltonian, which can be diagonalised as

$$\begin{aligned} A\hat{x}^2 + B\hat{p}^2 + C(\hat{x}\hat{p} + \hat{p}\hat{x}) &= B\left(\hat{p} + \frac{C}{B}\hat{x}\right)^2 + \left(A - \frac{C^2}{B}\right)\hat{x}^2 \\ &= B\hat{p}_{\Delta}^2 + \left(A - \frac{C^2}{B}\right)\hat{x}^2 , \end{aligned} \tag{289}$$

where the operator $\hat{p}_{\Delta} = \hat{p} + C\hat{x}/B$ has the standard commutation relation with $\hat{x}$ as well. Thus the operator in the exponential is just the usual Hamiltonian operator for a harmonic oscillator with mass of $1/(2B)$ and spring constant of $2(A - C^2/B)$ which we know has eigenvalues of $(n + 1/2)\omega$ where ω in this case is

$$\omega = 2\sqrt{AB - C^2} . \tag{290}$$

As shown in Sec. 4.7, the effective frequency ω is related to the squeezing parameter ξ via

$$\tanh^2 \xi = \exp\{-\omega\} . \tag{291}$$

Thus we get

$$Z = \mathrm{Tr}\left\{e^{-\omega(\hat{n}+1/2)}\right\} = \sum_{n=0}^{\infty} e^{-(2n+1)\sqrt{AB-C^2}} = \frac{2}{\sinh(\sqrt{AB - C^2})} . \tag{292}$$

This allows us to calculate the expectation values

$$\begin{aligned}\langle\hat{x}^2\rangle &= -\frac{\partial}{\partial A}\ln(Z) = \frac{B}{2\sqrt{AB-C^2}}\coth\sqrt{AB-C^2} = B\frac{\coth(\omega/2)}{\omega}\\ \langle\hat{p}^2\rangle &= -\frac{\partial}{\partial B}\ln(Z) = \frac{A}{2\sqrt{AB-C^2}}\coth\sqrt{AB-C^2} = A\frac{\coth(\omega/2)}{\omega}\\ \langle\hat{p}\hat{x}+\hat{x}\hat{p}\rangle &= -\frac{\partial}{\partial C}\ln(Z) = \frac{-C}{\sqrt{AB-C^2}}\coth\sqrt{AB-C^2} = -C\frac{\coth(\omega/2)}{\omega}\,, \end{aligned} \tag{293}$$

as well as the entropy of the reduced density matrix

$$S = -\mathrm{Tr}\left\{\hat{\rho}\ln\hat{\rho}\right\} = \ln(Z) - \frac{\omega}{Z}\frac{\partial Z}{\partial\omega} = \frac{\omega}{1-e^{-\omega}} - \ln(e^{\omega}-1)\,. \tag{294}$$

Since the whole system is in a pure state, this is exactly the entropy of entanglement between the $\hat{a}$ and $\hat{b}$ channels. For small ω, it diverges as $-\ln(\omega)$, and for large ω, it approaches zero. In this limit $\omega\uparrow\infty$, we get a pure state $\hat{\rho}\to|\psi\rangle\langle\psi|$. For the expectation values, this implies

$$\langle\hat{x}^2\rangle\langle\hat{p}^2\rangle - \langle\hat{p}\hat{x}+\hat{x}\hat{p}\rangle^2 = \frac{1}{4}\,. \tag{295}$$

If the above equality is not satisfied, the reduced density matrix has a thermal form with the above effective Hamiltonian. Consequently, a phase insensitive amplifier always increases the noise by adding thermal entropy to the output. We will see later that the Hawking process for a black hole can be regarded as such a phase insensitive amplifier, and that the thermal radiation can be regarded as the thermal noise output from such a phase insensitive amplifier – acting on the initial vacuum state.

Furthermore, the cosmological particle production during inflation, for example, can be regarded as the output from a phase sensitive amplifier (i.e., single-mode squeezing). The initial quantum vacuum fluctuations are amplified in one quadrature and de-amplified in the other – which leads directly to the anisotropies one sees in the Cosmic Microwave radiation spectrum.

However, one should not that the distinction between single-mode squeezing (phase sensitive amplification) and two-mode squeezing (phase insensitive amplification) is not always unique and may depend on the chosen modes. For example, one might diagonalise the $\boldsymbol{\xi}$-matrix (280) by introducing new modes $(\hat{a}\pm\hat{b})/\sqrt{2}$ for which we would have single-mode squeezing for two independent modes instead of two-mode squeezing. Such a rotation of modes corresponds to the operator $\hat{U}_{\boldsymbol{\chi}}$. For the example of cosmological particle production, one would have single-mode squeezing – i.e., the creation of pairs of particles in the same mode – for real mode functions $\cos(kx)$ and $\sin(kx)$. Using complex mode functions $\exp\{\pm ikx\}$, on the other hand, the same physical process is described by two-mode squeezing, i.e., one creates pairs of particles with opposite momenta.

4.9 Quantisation of Dirac Field

The quantisation of the Dirac field poses problems above those of the scalar or electromagnetic fields. In 3+1 dimensions, the spin-statistics theorem (see, e.g., [Streater & Wightman, 2000]) demands that spin-1/2 fields must obey the Pauli exclusion principle and must therefore be quantised with anti-commutators rather than commutators.

In order to illustrate why this is the case, let us first ignore this point and try to quantile the Dirac field in the same way as the scalar field in Sec. 4.2, for example. As usual, it is most convenient to start from the action (153) of the Dirac field

$$\mathcal{A} = \int d^4x \sqrt{-g} \left(i\bar{\psi}\gamma^\mu \nabla_\mu \psi + m\bar{\psi}\psi \right) . \tag{296}$$

Since this is a first-order action (containing only first-order derivatives), we have that $\Pi = i\sqrt{-g}\,\bar{\psi}\gamma^t$ is the momentum conjugate to ψ. To simplify the following analysis, let us assume that we have a static metric (where ∂_t corresponds to a Killing vector ξ^μ). Choosing the vier-bein appropriately (for this static metric) such that $e^{\mu=0}_{a=0} = \sqrt{g^{00}}$ while the mixed components vanish $e^{\mu=0}_{a>0} = 0$, we get $\Pi = i\sqrt{-g}\,\sqrt{g^{00}}\,\psi^\dagger$, i.e., Π contains the determinant of the spatial (three) metric.

Ignoring the fact that these are fermions, let us impose the usual (bosonic) canonical equal-time commutation relations

$$\begin{aligned}
\left[\hat{\psi}_a(t,\boldsymbol{r}), \hat{\psi}_b(t,\boldsymbol{r'})\right] &\stackrel{?}{=} 0\,,\ \left[\hat{\Pi}_a(t,\boldsymbol{r}), \hat{\Pi}_b(t,\boldsymbol{r'})\right] \stackrel{?}{=} 0 \rightsquigarrow \left[\hat{\psi}^\dagger_a(t,\boldsymbol{r}), \hat{\psi}^\dagger_b(t,\boldsymbol{r'})\right] \stackrel{?}{=} 0\,,\\
\left[\hat{\Pi}_a(t,\boldsymbol{r}), \hat{\psi}_b(t,\boldsymbol{r'})\right] &\stackrel{?}{=} i\delta_{ab}\,\delta^3(\boldsymbol{r},\boldsymbol{r'})\,,
\end{aligned} \tag{297}$$

where a and b label the components of the bi-spinors ψ and Π. Note that, using this convention, the Dirac distribution $\delta^3(\boldsymbol{r},\boldsymbol{r'})$ does not contain the $1/\sqrt{-g}$ factor, since this is already included in Π.

For the scalar field, one can use the pseudo-norm to classify the modes of the field, with positive pseudo-norm modes to be treated differently from negative. For the Dirac equation, on the other hand, the norm of all fields is positive (on the classical level). As usual, the norm can be obtained from the conserved current J^μ via

$$(\psi|\psi) = \int d\Sigma_\mu\, J^\mu(\psi) = \int d\Sigma_\mu\, \bar{\psi}\gamma^\mu\psi = \int d\Sigma_\mu\, e^\mu_a\, \bar{\psi}\gamma^a\psi\,. \tag{298}$$

This form can be generalised to the conserved inner product by inserting two different solutions ψ_1 and ψ_2 of the Dirac equation. Choosing the slicing Σ (and the vier-bein e^μ_a) appropriately, this simplifies to

$$(\psi_1|\psi_2) = \int d\Sigma_\mu\, \bar{\psi}_1\gamma^\mu\psi_2 = \int d^3x\,\sqrt{-g}\,e^0_0\,\psi^\dagger_1\psi_2 = -i\int d^3x\,\Pi_1\psi_2\,. \tag{299}$$

Since $(\psi_1|\psi_2)$ has the standard properties of a scalar product (and is conserved by the time-evolution), we may always find a complete and orthonormal set ψ_I of solutions of the Dirac equation with $(\psi_I|\psi_J) = \delta(I, J)$. Expanding the field operator in terms of these solutions via

$$\hat{a}_I = \left(\psi_I|\hat{\psi}\right) \rightsquigarrow \hat{\psi}(t, \boldsymbol{r}) = \sum\!\!\!\!\!\!\int_I \hat{a}_I\, \psi_I(t, \boldsymbol{r}) \,, \tag{300}$$

the naive commutation relations (297) imply, using the property (299),

$$[\hat{a}_I, \hat{a}_J] \stackrel{?}{=} \left[\hat{a}_I^\dagger, \hat{a}_J^\dagger\right] \stackrel{?}{=} 0 \,, \quad \left[\hat{a}_I, \hat{a}_J^\dagger\right] \stackrel{?}{=} \delta(I, J) \,, \tag{301}$$

showing that the $\hat{a}_I^\dagger$ and $\hat{a}_J$ are bosonic creation and annihilation operators. However, this quantisation procedure yields an energy operator which is not bounded from below and is therefore highly problematic: The time-like Killing vector ξ^μ allows us to obtain a conserved energy

$$E = \int d\Sigma_\mu\, T^{\mu\nu}\, \xi_\nu = \int d^3x\, \sqrt{-g}\, T^0_0 = \frac{i}{2} \int d^3x\, \sqrt{-g} \left(\bar{\psi}\, \gamma^t\, \overset{\leftrightarrow}{\nabla}_t\, \psi\right) \,. \tag{302}$$

Assuming a vanishing vector potential $A_\mu = 0$ (and using the natural choice of the vierbein for a static metric), we have $\Gamma_t = \Gamma_\xi = 0$ and thus $\nabla_t = \partial_t$. In addition, as explained in Section 3.8, the Killing vector allows us to make the separation ansatz

$$\psi_I(t, \boldsymbol{r}) = \exp\{-i\omega_I t\}\psi_I(\boldsymbol{r}) \,\leftrightarrow\, \pounds_\xi \psi_I = \partial_t \psi_I = \nabla_t \psi_I = -i\omega_I \psi_I \,. \tag{303}$$

Inserting the expansion (300) together with this ansatz (303) into equation (302) gives

$$\begin{aligned} \hat{E} &= \sum\!\!\!\!\!\!\int_{I,J} \hat{a}_I^\dagger \hat{a}_J\, \frac{\omega_I + \omega_J}{2} \int d^3x\, \sqrt{-g}\, e^0_0\, \psi_I^\dagger(\boldsymbol{r})\psi_J(\boldsymbol{r}) = \sum\!\!\!\!\!\!\int_{I,J} \hat{a}_I^\dagger \hat{a}_J\, \frac{\omega_I + \omega_J}{2}\, (\psi_I|\psi_J) \\ &= \sum\!\!\!\!\!\!\int_I \omega_I \hat{a}_I^\dagger \hat{a}_I = \sum\!\!\!\!\!\!\int_I \omega_I \hat{n}_I \,. \end{aligned} \tag{304}$$

Now, as shown in Section 3.9, one has negative energy modes in this case (again on the classical level): For each solution ψ_+ with positive energy, there is another solution $\psi_- = C\psi_+^*$ with negative energy and vice versa. Therefore, for each positive eigenvalue $\omega_I > 0$, there must be a corresponding negative energy solution $\omega_J = -\omega_I < 0$. For bosons, we may occupy these negative energy modes with more and more particles – thus making the energy arbitrarily negative. This would indicate that the theory is highly unstable: Adding the interaction to the electromagnetic field, for example, the particles could decay to lower and lower negative energy levels via emitting more and more photons.

In order to cure this problem, we have to impose fermionic anti-commutation relations instead of the usual bosonic commutators (297)

$$\begin{aligned}\left\{\hat\psi_a(t,\boldsymbol{r}),\hat\psi_b(t,\boldsymbol{r'})\right\} &= \hat\psi_a(t,\boldsymbol{r})\,\hat\psi_b(t,\boldsymbol{r'})+\hat\psi_b(t,\boldsymbol{r'})\,\hat\psi_a(t,\boldsymbol{r})=0\,,\\ \left\{\hat\Pi_a(t,\boldsymbol{r}),\hat\Pi_b(t,\boldsymbol{r'})\right\} &= 0 \leadsto \left\{\hat\psi_a^\dagger(t,\boldsymbol{r}),\hat\psi_b^\dagger(t,\boldsymbol{r'})\right\}=0\,,\\ \left\{\hat\Pi_a(t,\boldsymbol{r}),\hat\psi_b(t,\boldsymbol{r'})\right\} &= i\delta_{ab}\,\delta^3(\boldsymbol{r},\boldsymbol{r'})\,.\end{aligned} \tag{305}$$

If we now employ the same expansion as in (300), we find

$$\{\hat a_I,\hat a_J\}=\left\{\hat a_I^\dagger,\hat a_J^\dagger\right\}=0\,,\;\left\{\hat a_I^\dagger,\hat a_J\right\}=\delta(I,J)\,, \tag{306}$$

showing that the $\hat a_I^\dagger$ and $\hat a_J$ have the properties of fermionic creation and annihilation operators (as expected).

Now one could try to define the vacuum state as $\forall_I \hat a_I\,|0\rangle=0$. However, that definition is also problematic: With the same arguments as used after Eq. (304), one can show that this vacuum would not be the state with lowest energy – one could lower the energy by creating particles in a mode with negative energy $\omega_I<0$.

This motivates the following Dirac sea construction. Let us split the modes ψ_I into two sets, the ones ψ_{I^+} with positive energy $\omega_{I^+}>0$ and the ones ψ_{I^-} with negative energy $\omega_{I^-}>0$. Here, we neglect possible zero-modes with $\omega_I=0$ which are constant in time and do neither contribute to the dynamics nor to the energy. Now we use the particle-hole duality for fermions and exchange $\hat a_I^\dagger\leftrightarrow\hat a_I$ for those modes I^- with $\omega_{I^-}>0$. Consequently, Eq. (300) is replaced by the ansatz

$$\hat\psi(t,\boldsymbol{r})=\sum\!\!\!\!\!\!\int_{I^+}\hat a_{I^+}\psi_{I^+}(t,\boldsymbol{r})+\sum\!\!\!\!\!\!\int_{I^-}\hat b_{I^-}^\dagger\psi_{I^-}(t,\boldsymbol{r})\,. \tag{307}$$

This leaves the fermionic anti-commutation relations (306) invariant

$$\begin{aligned}\{\hat a_I,\hat a_J\}=\left\{\hat a_I^\dagger,\hat a_J^\dagger\right\}=\left\{\hat b_I,\hat b_J\right\}=\left\{\hat b_I^\dagger,\hat b_J^\dagger\right\} &= 0\,,\\ \left\{\hat a_I,\hat b_J\right\}=\left\{\hat a_I^\dagger,\hat b_J^\dagger\right\}=\left\{\hat a_I^\dagger,\hat b_J\right\}=\left\{\hat a_I,\hat b_J^\dagger\right\} &= 0\,,\\ \left\{\hat a_I^\dagger,\hat a_J\right\}=\left\{\hat b_I^\dagger,\hat b_J\right\} &= \delta(I,J)\,.\end{aligned} \tag{308}$$

Note this this duality does not work for bosons (as we would get additional minus signs). With this ansatz (307), the energy becomes

$$\hat{E} = \sum\!\!\!\!\!\!\!\int_{I^+} \omega_{I^+}\hat{a}^\dagger_{I^+}\hat{a}_{I^+} + \sum\!\!\!\!\!\!\!\int_{I^-} \omega_{I^-}\hat{b}_{I^-}\hat{b}^\dagger_{I^-} = \sum\!\!\!\!\!\!\!\int_{I^+} \omega_{I^+}\hat{n}_{I^+} + \sum\!\!\!\!\!\!\!\int_{I^-} |\omega_{I^-}|\,\hat{n}_{I^-} + E_\infty\,, \qquad (309)$$

which is positive definite apart from a divergent c-number – the zero-point energy E_∞. Now we may define the vacuum as the state with lowest energy via

$$\forall_{I^+}\hat{a}_{I^+}|0\rangle = 0\,,\ \forall_{I^-}\hat{b}_{I^-}|0\rangle = 0\,. \qquad (310)$$

Intuitively, this vacuum corresponds to the state where all the levels with a negative energy are filled while all the positive levels are empty. Due to the Pauli principle, it is not possible to gain energy since it the negative energy levels cannot be occupied twice. However, by spending some energy, it is possible to extract a particle from this Dirac sea – thereby leaving behind a hole, which can be interpreted as the corresponding anti-particle (for electrons, these would be positrons). Therefore, the operation $\psi_- = C\psi_+^*$ can be interpreted as charge conjugation $\mathcal{C}$ (i.e., the $\mathcal{C}$ in the $\mathcal{CPT}$ theorem) which transforms particles into anti-particles and vice versa. To make this more explicit, let us insert the expansion (307) into the norm

$$\left(\hat{\psi}|\hat{\psi}\right) = \sum\!\!\!\!\!\!\!\int_{I^+} \hat{a}^\dagger_{I^+}\hat{a}_{I^+} + \sum\!\!\!\!\!\!\!\int_{I^-} \hat{b}_{I^-}\hat{b}^\dagger_{I^-} = \sum\!\!\!\!\!\!\!\int_{I^+} \hat{n}_{I^+} - \sum\!\!\!\!\!\!\!\int_{I^-} \hat{n}_{I^-} + Q_\infty\,. \qquad (311)$$

As one would expect from the derivation of the inner product (299) via the $U(1)$ symmetry (Noether theorem), this gives the operator of the total charge $\hat{Q} = \left(\hat{\psi}|\hat{\psi}\right) - Q_\infty$ after subtracting the infinite charge Q_∞ of the Dirac sea (similar to the zero-point energy E_∞ of the Dirac sea).

In summary, on the classical level, the norm is always positive $(\psi|\psi) > 0$, while the energy can be positive as well as negative. In contrast, on the quantum level (after renormalisation, i.e., subtracting E_∞ and Q_∞), the energy is positive but the norm (i.e., charge) can be positive as well as negative. This inversion is facilitated by the fermionic anti-commutation relations – which is a basic ingredient for the spin-statistics theorem.

Note that the above vacuum definition (310) crucially depends on the Killing vector and is therefore not unique in general. If there is no Killing vector or more than one (time-like) Killing vector (e.g., the Minkowski and Rindler time), one cannot define a unique vacuum

– which facilitates phenomena like particle creation (such as Hawking radiation). The naive vacuum definition $\forall_I \hat{a}_I |0\rangle = 0$ would be unique and covariant, but it is unphysical (e.g., there would be no particle creation at all). Particle creation such as Hawking radiation requires the Dirac sea (and thus the fermionic commutation relations). In the bosonic case, the particle-hole duality $\hat{a}_I^\dagger \leftrightarrow \hat{a}_I$ does not apply and there is no state $|0\rangle$ which obeys $\hat{a}_I^\dagger |0\rangle = 0$.

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Let us divide the set of all solutions to the Dirac equations into two sub-sets. The first sub-set will be some arbitrary choice of "half" the modes with have

The second set will be taken to be the complex conjugate $Q\psi^*$ of the ones in the first set. Let ψ_j be an orthonormal set of the first positive energy modes, and $Q\psi_i^*$ is the corresponding negative set.

Choose the two sets so that all of the second set are orthogonal to all of the first.

To implement Pauli's idea, we write the quantum field ψ as

$$\Psi = \sum_j a_j\psi_j + b_j^\dagger Q\psi_j^* \tag{312}$$

With

$$\bar{\Psi} = \sum_j a_j^\dagger\bar{\psi}_j + b_j\bar{\psi}_j^* Q \tag{313}$$

these two fields will obey the commutation relations because of the assumed completeness of this set of modes if we take

$$\{a_j^\dagger, a_k\} = \{b_j^\dagger, b_k\} = \delta_{ij} \tag{314}$$

and all other anti-commutators being zero.

Note that because of the anti-commutators, it would be perfectly consistent to take

$$\Psi = \sum_j (a_j\psi_j + b_j Q\psi_j^*) \tag{315}$$

This use of the $b^\dagger$ for the complex conjugated modes (charge conjugation as it is usually called) is the Dirac "filled sea" principle in operation.

One can now define a "vacuum" state by

$$a_j|0\rangle = b_j|0\rangle = 0 \tag{316}$$

Note that this state has nothing to do with any concept of energy, but is defined purely using the commutation relations of the field and the norm.

Of course this has been a bit too arbitrary, since there is a danger that a theory containing such a field would be unstable. We can now bring in the total energy function

$$E = \bar{\Psi}\gamma^t\partial_t\psi + \tag{317}$$

Again,just as for the scalar field, there is no natural vacuum state for this field. If there exists a time-like Killing vector which is future directed everywhere along a Cauchy surface, then one could take as the natural vacuum state the minimum energy defined by this Killing vector $H = \int \zeta_\mu T^{\mu t}\sqrt{(}-g)d^3x$ where $t = \text{const}$ is assumed to be the Cauchy surface.

The field Ψ is promoted to a field of quantum operators, one at each space-time point. From the action, $\sqrt{-g}i\bar{\Psi}\gamma^t$ is the momentum conjugate to Ψ. However, because of the negativity of the energy operator, we define the field by anti-commutators rather than commutators.

$$\{i\sqrt{-g}\dagger\Psi\gamma^t, \Psi(t,x)\} = iI\delta(x,x') \tag{318}$$
$$\{\Psi(t,x), \Psi(t,x')\} = \{\bar{\Psi}(t,x), \bar{\Psi}(t,x')\} = 0 \tag{319}$$

If $\psi_j(t,x)$ are a set of orthonormal (under the norm

$$< \tilde{\psi}, \psi >= \int \sqrt{-g}\bar{\tilde{\psi}}\gamma^t\psi d^3x) \tag{320}$$

solutions to the classical Dirac, such that $Q\psi_j^*$ and ψ_j form a complete set of solutions. Here Q is defined by

$$-Q(\gamma^a)^*Q = \gamma^a \tag{321}$$
$$Q^2 = I \tag{322}$$

so that $Q\psi^*$ is a solution to the Dirac equation, if ψ is. with our choice of γ^a, we can choose $Q = \gamma^0$.

We can then write

$$\Psi(t,x) = \sum_j (a_j\psi_j(t,x) + b_j^\dagger Q\psi_j^*(t,x)) \tag{323}$$

in terms of this complete set of solutions, where

$$\{a_j, a_k\} = \{b_j, b_k\} = \{a_j, b_j\} = 0 \tag{324}$$
$$\{a_j, b_k^\dagger\} = 0\{a_j, a_k^\dagger\} =< b_j, b_k^\dagger\} = \delta_{jk} \tag{325}$$

(where the † here is with respect to the operator Hilbert space, not the 4 dimensional function space of the field.) As usual one can define a "vacuum" state with respect to this set of modes by

$$a_j|0\rangle = b_j|0\rangle = 0 \tag{326}$$

Note that there is essentially no restriction on the set of modes ψ_j one can use, unlike in the scalar case where one must use a set of positive norm modes. In the case of the Dirac modes, all have positive norm.
There may however be physical restrictions on the choice of modes. In particular, the "Energies" of these modes is not necessarily positive.
The energy, defined as the integral over a space-like surface of the energy current of the Stress energy tensor is an ill defined concept. Because the energy momentum tensor is a tensor, and not a vector, the energy current is in general not conserved.

$$\mathcal{J}^{\mu} = n_{\alpha} T^{\alpha\mu} \tag{327}$$

is not conserved unless

$$\nabla_{\mu} n_{\alpha} T^{\alpha\mu} = 0 \tag{328}$$

Ie, unless n_{α} is a Killing vector. This means that the total energy is a function of both the vector n chosen to define the energy current, and on the hypersurface over which one integrates that current. This simplest case is to choose n_{α} to be a time-like Killing vector and choose the hypersurface to be a constant t hypersurface, where t is the integration parameter along the integral curves of n^{α}. In this t coordinate system, the metric is time independent, and the Killing vector equation is

$$\pounds_n \psi = \partial_t \psi \tag{329}$$

One now chooses the modes associated with the a operators to be the "negative" eigenvalue modes of the Lee derivative along n.

$$\pounds_n \psi_{\omega} = -i\omega\psi_{\omega} \tag{330}$$

(ie, the modes ψ_j can be written as sums only over the positive values of ω

$$\psi_i = \int_{\omega} > 0 \alpha_{\omega k} \psi_{\omega k} d\omega dk \tag{331}$$

where k are are the extra labels needed to resolve any degeneracy in the modes of frequency ω. This corresponds most closely to the "filled negative energy sea" of the Dirac
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[Ref: PCT, Spin, Statistics, and all that– Streater and A Wightman)]
Raymond F. Streater, Arthur S. Wightman PCT, Spin and Statistics, and All That (Princeton University Press, Princeton, 2000)

4.10 Majorana Field in Curved Space-time

The Majorana field theory is closely related to the Dirac field theory discussed in the previous Section – but whereas the Dirac field has two different kinds of particle, the particles and their anti-particles, in the Majorana case, there is only one. The particle is its own anti-particle.

The Majorana field can most easily be described using the Majorana representation (178) of the γ matrices, in which all of the γ^a are purely imaginary. Let us start in flat space-time for simplicity. In this case the field equations for the Dirac spinor become equations with real coefficients

$$i\gamma^a \partial_a \psi + m\psi = 0 \,. \tag{332}$$

Because the gamma matrices are all purely imaginary, the complex conjugate ψ^* is also a solution, as are the real $\Re(\psi) = (\psi + \psi^*)/2$ and imaginary $\Im(\psi) = (\psi - \psi^*)/2i$ parts of the wave function. We can thus demand that the solution be real $\psi^* = \psi$.

In curved space-time, we have

$$\gamma^\mu = e^\mu_a \gamma^a \,, \tag{333}$$

and since the e^μ_a are all real, the γ^μ are again purely imaginary matrices.

But one should remember that we also have to replace the ∂_a by the covariant derivative ∇_μ containing the spin connections Γ_μ, see Eq. (139). The metric part of the spin connections Γ_μ is given by $G_{\alpha\beta\mu}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)$ where the coefficients $G_{\alpha\beta\mu}$ (derivatives of the metric and the e^μ_a) are purely real. The commutator $\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha$ of two purely imaginary matrices is also purely real. Thus the metric part of the spin connections Γ_μ is real and can be included for the real Majorana field $\psi^* = \psi$.

However, the $U(1)$ part of the spin connections Γ_μ in Eq. (139) is given by $iA_\mu \mathbf{1}$ which is purely imaginary. Thus, adding this part would spoil the above reality condition and mix real and imaginary parts of ψ. As a consequence, if we demand that we have a purely real field $\psi^* = \psi$, we cannot include this term – i.e., we cannot couple the Majorana field $\psi^* = \psi$ to a vector field A_μ via the usual minimal coupling procedure $\nabla_\mu \to \nabla_\mu + iA_\mu$, which is consistent with the absence of the $U(1)$-symmetry $\psi \to e^{i\phi}\psi$ for Majorana fields $\psi^* = \psi$. This is a quite reasonable result since one would expect that particles (e.g., electrons) react to the vector (e.g., electric) field A_μ in the opposite way to anti-particles (e.g., positrons). Thus, if the particles are their own anti-particles, they should not react to A_μ at all (at least in the absence of other fields).

In order to quantise the Majorana field, let us try to proceed in a way analogous to the previous Section and start from the action (153)

$$\mathcal{A} = \int d^4x \sqrt{-g} \left(i\bar{\psi}\gamma^\mu\nabla_\mu\psi + m\bar{\psi}\psi \right) . \tag{334}$$

One might object that this action is not suitable for real fields since the term $\psi^T\dot{\psi}$, for example, is a total derivative $\partial_t(\psi^T\psi) = \psi^T\dot{\psi} + \dot{\psi}^T\psi = 2\psi^T\dot{\psi}$. One possible reply could be that the last step in this equation is correct for c-number fields, but not if one already envisages anti-commuting (Grassmann type) fields.
However, we shall not dwell on this issue here and directly go to field operators $\psi \to \hat{\Psi}$. In this case, the reality of the Majorana field ψ implies that each component (labelled by a, b, c etc.) of the field operator $\hat{\Psi}$ is self-adjoint

$$\hat{\Psi}_a = \hat{\Psi}_a^\dagger . \tag{335}$$

For simplicity, we again assume a static metric with the same vier-bein as in the previous Section, i.e., $e^{\mu=0}_{a=0} = \sqrt{g^{00}}$ etc. The momentum density $\hat{\Pi} = i\sqrt{-g}\sqrt{g^{00}}\,\hat{\Psi}^T$ is then anti-self-adjoint $\hat{\Pi}_a = -\hat{\Pi}_a^\dagger$ and we get the anti-commutation relations

$$\left\{ \hat{\Psi}_a(t,\boldsymbol{r}), \hat{\Psi}_b(t,\boldsymbol{r}') \right\} = \hat{\Psi}_a(t,\boldsymbol{r})\,\hat{\Psi}_b(t,\boldsymbol{r}') + \hat{\Psi}_b(t,\boldsymbol{r}')\,\hat{\Psi}_a(t,\boldsymbol{r}) = \delta_{ab}\,\delta^3(\boldsymbol{r},\boldsymbol{r}') . \tag{336}$$

Introducing the same (conserved) scalar product as before

$$(\psi_1|\psi_2) = \int d\Sigma_\mu\,\bar{\psi}_1\gamma^\mu\psi_2 = \int d^3x\,\sqrt{-g}\,e^0_0\,\psi_1^T\psi_2 , \tag{337}$$

we may expand the field operator $\hat{\Psi}$ into a complete and orthonormal (with respect to this scalar product) set of real solutions $\psi_I(t,\boldsymbol{r}) = \psi_I^*(t,\boldsymbol{r})$ of the Dirac equation

$$\hat{m}_I = \left(\psi_I|\hat{\Psi}\right) \rightsquigarrow \hat{\Psi}(t,\boldsymbol{r}) = \sum\!\!\!\!\!\!\int_I \hat{m}_I\,\psi_I(t,\boldsymbol{r}) . \tag{338}$$

The Majorana mode operators $\hat{m}_I$ are then self-adjoint $\hat{m}_I = \hat{m}_I^\dagger$ and obey the anti-commutation relations

$$\{\hat{m}_I, \hat{m}_J\} = \delta(I,J) . \tag{339}$$

Note that, in contrast to Dirac fermions with $\hat{a}_I^2 = 0$, we have $\hat{m}_I^2 = 1/2$ in this case (for discrete modes I). As a result, occupying one mode with two Majorana fermions is not forbidden (as in the Dirac case) but just reproduces the original state (up to a factor).

As an interesting consequence of the aforementioned properties of the Majorana field, we note that the expression for the conserved charge analogous to the Dirac case

$$\hat{Q} = \int d\Sigma_\mu \, \hat{\bar{\Psi}} \gamma^\mu \hat{\Psi} = \left(\hat{\Psi}|\hat{\Psi}\right) = \sum\!\!\!\!\!\!\int_I \hat{m}_I^2 = Q_\infty \,, \tag{340}$$

just gives an infinite c-number and hence does not provide any information about the actual number of Majorana fermions.
Now let us consider the Hamiltonian for the Majorana field, which can be obtained from the action (334) in the usual manner

$$\hat{H} = -\int d^3x \, \sqrt{-g} \left(\sum_{\mu=1}^{3} i \hat{\bar{\Psi}} \gamma^\mu \nabla_\mu \hat{\Psi} + m \hat{\bar{\Psi}} \hat{\Psi} \right) = i \left(\hat{\Psi} \middle| \mathcal{D} \middle| \hat{\Psi} \right) \,, \tag{341}$$

where $\mathcal{D}$ is a purely real matrix spatial differential operator, which is, however, anti-self-adjoint with respect to the above scalar product. Using the Dirac equation, this gives the same expression as the energy in Eq. (302).
If we now insert the mode expansion (338), we find that

$$\hat{H} = i \sum\!\!\!\!\!\!\int_{IJ} \hat{m}_I \, \hat{m}_J \, (\psi_I | \, \mathcal{D} \, | \psi_J) = i \sum\!\!\!\!\!\!\int_{IJ} \hat{m}_I \, \hat{m}_J \, \mathcal{M}_{IJ} = i \, \hat{\boldsymbol{m}} \cdot \boldsymbol{\mathcal{M}} \cdot \hat{\boldsymbol{m}} \,, \tag{342}$$

where $\boldsymbol{\mathcal{M}}$ is a real anti-symmetric matrix $\mathcal{M}_{IJ} = -\mathcal{M}_{JI} \in \mathbb{R}$. Such a matrix can be diagonalised and possesses a complete set of eigenvectors $\boldsymbol{\mathcal{M}} \cdot \boldsymbol{u}_\Lambda = i\omega_\Lambda \boldsymbol{u}_\Lambda$ with purely imaginary eigenvalues $\pm i\omega_\Lambda$ which occur in pairs, i.e., for each eigenvector $\boldsymbol{u}_\Lambda$ with eigenvalue $+i\omega_\Lambda$ where $\omega_\Lambda \geq 0$, there is another one $\boldsymbol{u}_\Lambda^*$ with the opposite eigenvalue $-i\omega_\Lambda$. For simplicity, we again discard zero modes with $\omega_\Lambda = 0$ in the following (as in the previous Section) since they do not contribute to the energy or the dynamics.
Now we may introduce new operators via

$$\hat{c}_\Lambda = \hat{\boldsymbol{m}} \cdot \boldsymbol{u}_\Lambda \,, \quad \hat{c}_\Lambda^\dagger = \hat{\boldsymbol{m}} \cdot \boldsymbol{u}_\Lambda^* \,, \tag{343}$$

which obey the usual fermionic anti-commutation relations as in Eq. (306)

$$\{\hat{c}_\Lambda, \hat{c}_\Omega\} = \left\{\hat{c}_\Lambda^\dagger, \hat{c}_\Omega^\dagger\right\} = 0 \,, \; \left\{\hat{c}_\Lambda^\dagger, \hat{c}_\Omega\right\} = \delta(\Lambda, \Omega) \,, \tag{344}$$

due to the ortho-normality of the eigenvectors $\boldsymbol{u}_\Lambda$ and $\boldsymbol{u}_\Lambda^*$ with respect to the (complex) scalar product, i.e., $\boldsymbol{u}_\Lambda \cdot \boldsymbol{u}_\Omega = \boldsymbol{u}_\Lambda^* \cdot \boldsymbol{u}_\Omega^* = 0$ and $\boldsymbol{u}_\Lambda^* \cdot \boldsymbol{u}_\Omega = \delta(\Lambda, \Omega)$. In terms of the Majorana field operator, this implies

$$\hat{\Psi}(t,\boldsymbol{r}) = \sum\!\!\!\!\!\!\!\int_I \hat{m}_I\,\psi_I(t,\boldsymbol{r}) = \sum\!\!\!\!\!\!\!\int_\Lambda \left[\hat{c}_\Lambda\,\psi_\Lambda(t,\boldsymbol{r}) + \hat{c}^\dagger_\Lambda\,\psi^*_\Lambda(t,\boldsymbol{r})\right], \tag{345}$$

where ψ_Λ and ψ^*_Λ are the projections of the original real mode functions ψ_I onto the complex eigenvectors $\boldsymbol{u}_\Lambda$ and $\boldsymbol{u}^*_\Lambda$. Note that, in contrast to the Dirac field (307), we have the same operator in the two terms $\hat{c}_\Lambda\,\psi_\Lambda$ and $\hat{c}^\dagger_\Lambda\,\psi^*_\Lambda$ which means that there is no real difference between particles and anti-particles.
Using the completeness of the eigenvectors $\boldsymbol{u}_\Lambda$ and $\boldsymbol{u}^*_\Lambda$, we may invert (343)

$$\hat{\boldsymbol{m}} = \sum\!\!\!\!\!\!\!\int_\Lambda \left(\hat{c}_\Lambda\,\boldsymbol{u}^*_\Lambda + \hat{c}^\dagger_\Lambda\,\boldsymbol{u}_\Lambda\right). \tag{346}$$

Inserting this expression into the Hamiltonian and using $\boldsymbol{\mathcal{M}}\cdot\boldsymbol{u}_\Lambda = i\omega_\Lambda\boldsymbol{u}_\Lambda$ as well as the ortho-normality of the eigenvectors $\boldsymbol{u}_\Lambda$ and $\boldsymbol{u}^*_\Lambda$, we find

$$\hat{H} = 2\sum\!\!\!\!\!\!\!\int_\Lambda \omega_\Lambda\hat{c}^\dagger_\Lambda\hat{c}_\Lambda + E_\infty\,, \tag{347}$$

where E_∞ is a divergent zero-point energy which is a c-number. In contrast to the Dirac case, here the ω_Λ are all positive (or at least non-negative), so there is no Dirac sea and the ground state (i.e., the vacuum) is simply given by

$$\forall_\Lambda\,\hat{c}_\Lambda\,|0\rangle = 0\,. \tag{348}$$

Note, however, that in terms of the Majorana operators $\hat{m}_I$, this is a strongly correlated state. To see that, let us introduce another set of Majorana operators via

$$\hat{m}^\Re_\Lambda = \frac{\hat{c}_\Lambda + \hat{c}^\dagger_\Lambda}{\sqrt{2}} = \sqrt{2}\,\hat{\boldsymbol{m}}\cdot\Re(\boldsymbol{u}_\Lambda)\,, \quad \hat{m}^\Im_\Lambda = \frac{\hat{c}_\Lambda - \hat{c}^\dagger_\Lambda}{\sqrt{2}\,i} = \sqrt{2}\,\hat{\boldsymbol{m}}\cdot\Im(\boldsymbol{u}_\Lambda)\,, \tag{349}$$

which do also satisfy the Majorana commutation relations (339). In terms of these operators, the condition (348) reads $(\hat{m}^\Re_\Lambda + i\hat{m}^\Im_\Lambda)\,|0\rangle = 0$. As a result, there is a strong correlation between these Majorana particles, even in the vacuum state

$$\langle 0|\,\hat{m}^\Re_\Lambda\hat{m}^\Im_\Omega\,|0\rangle - \langle 0|\,\hat{m}^\Re_\Lambda\,|0\rangle\,\langle 0|\,\hat{m}^\Im_\Omega\,|0\rangle = \langle 0|\,\hat{m}^\Re_\Lambda\hat{m}^\Im_\Omega\,|0\rangle = \frac{1}{2i}\,\delta(\Lambda,\Omega)\,. \tag{350}$$

Intuitively speaking, these Majorana fermions always come in pairs.
Note that the above arguments, such as the self-adjointness of the field operator components $\hat{\Psi}_a = \hat{\Psi}_a^\dagger$ and their anti-commutator (336) are restricted to the Majorana representation (178) of the γ matrices. In other representations, they have to be modified. To see this, let us consider a unitary rotation $S^\dagger = S^{-1}$ in spinor space

$$\psi \to S\psi\,. \tag{351}$$

Demanding that $\bar{\psi}\psi$ and $\bar{\psi}\gamma^\mu\psi$ should be invariant, we get the transformation laws

$$G \to SGS^\dagger\,, \quad \gamma^\mu \to S\gamma^\mu S^\dagger\,. \tag{352}$$

Then, if we start in the Majorana representation (178) with $(\gamma^\mu)^* = -\gamma^\mu$, we get for the transformed gamma matrices

$$SS^T(\gamma^\mu)^* S^* S^\dagger = -\gamma^\mu\,. \tag{353}$$

Thus, for the new representation, the (charge conjugation) matrix C reads

$$C = SS^T = (S^*S^\dagger)^\dagger \rightsquigarrow C(\gamma^\mu)^* C^\dagger = -\gamma^\mu\,, \tag{354}$$

which is indeed unitary $C^\dagger C = S^*S^\dagger SS^T = S^*S^T = (SS^\dagger)^* = \mathbf{1}$ but not self-adjoint in general. In this general representation, the original Majorana reality condition $\psi = \psi^*$ transforms into $\psi = C\psi^*$ and thus the property (335) now becomes

$$\hat{\Psi}_a = \sum_b C_{ab}\hat{\Psi}_b^\dagger\,. \tag{355}$$

As a result, the mode expansion (345) reads now

$$\hat{\Psi}(t,\boldsymbol{r}) = \not\!\!\!\!\!\sum_\Lambda \left[\hat{c}_\Lambda\,\psi_\Lambda(t,\boldsymbol{r}) + \hat{c}_\Lambda^\dagger\,C\psi_\Lambda^*(t,\boldsymbol{r})\right]\,, \tag{356}$$

but of course, this change of the representation does not affect any of the physical properties.
Finally, let us discuss the phenomenon of particle creation for the Majorana case. Let us assume that the energy measured by one observer is given by the Hamiltonian $\hat{H}_{\rm in}$ containing the matrix $\boldsymbol{\mathcal{M}}_{\rm in}$ which can be diagonalised via the operators $\hat{c}_\Lambda$ and $\hat{c}_\Lambda^\dagger$. This observer would define the vacuum state by (348). Another observer would employ a different Hamiltonian $\hat{H}_{\rm out}$ with the matrix $\boldsymbol{\mathcal{M}}_{\rm out}$, which can be diagonalised via the

operators $\hat{d}_\Lambda$ and $\hat{d}_\Lambda^\dagger$. In general, these two sets of operators are then related to each other via a Bogoliubov transformation

$$\hat{d}_\Omega = \sum\!\!\!\!\!\!\!\int_\Lambda \left(\alpha_{\Omega\Lambda}\hat{c}_\Lambda + \beta_{\Omega\Lambda}\hat{c}_\Lambda^\dagger\right) . \tag{357}$$

Unless all the $\beta_{\Omega\Lambda}$ are zero, the two observers will not agree on the same vacuum state and there will be particle creation. Note that the above transformation just corresponds to a simple real rotation in the space of Majorana operators $\hat{\boldsymbol{m}} \to \boldsymbol{\mathcal{D}} \cdot \hat{\boldsymbol{m}}$ with a real orthogonal matrix $\boldsymbol{\mathcal{D}}^T = \boldsymbol{\mathcal{D}}^{-1}$ such that the phenomenon of particle creation is not so apparent in this representation.

There is also another interesting observation: In contrast to bosons, particle creation for fermions cannot occur in a single mode only, unless the role of particles and holes (anti-particles) in this mode is reversed completely. The reason is that the diagonal Bogoliubov transformation $\hat{d}_\Lambda = \alpha_\Lambda \hat{c}_\Lambda + \beta_\Lambda \hat{c}_\Lambda^\dagger$ is not compatible with the fermionic anti-commutation relations unless either α_Λ or β_Λ vanishes. Thus, apart from this exceptional case, fermionic particle creation requires involving at least two modes. Another way of understanding this is to consider the general time-evolution operator for linear fermionic fields

$$|\psi\rangle_{\text{out}} = \hat{U}\,|0\rangle = \exp\left\{ i \sum\!\!\!\!\!\!\!\int_{\Lambda\Omega} \left(\xi_{\Omega\Lambda}\hat{c}_\Lambda^\dagger\hat{c}_\Omega^\dagger + \chi_{\Omega\Lambda}\hat{c}_\Lambda^\dagger\hat{c}_\Omega + \text{h.c.}\right)\right\} |0\rangle\ . \tag{358}$$

In contrast to bosons, the diagonal elements of $\xi_{\Omega\Lambda}$ do not contribute for fermions. The off-diagonal elements of $\xi_{\Omega\Lambda}$ imply the creation of fermions pairs by acting $\hat{c}_\Lambda^\dagger\hat{c}_\Omega^\dagger$ on the vacuum. Since each of these operators (e.g., $\hat{c}_\Lambda^\dagger$) consists of two Majorana operators ($\hat{m}_\Lambda^\Re$ and $\hat{m}_\Lambda^\Im$), we see that four Majorana particles are involved in that process.

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being sums of products with real coefficients (derivatives of the real e^μ_a, with $[\gamma^a,\gamma^b]$ which are real matrices. Ie, the spin connection matrices are real, leaving the Dirac equation as a real equation. Thus the above argument in flat spacetime goes through also for a curved spacetime, with a similar ambiguity (and restrictions on that ambiguity) in the choice of the basis solutions.
,i.e., as quantum operators,
no coupling to A_μ

$$\Psi_A^\dagger = \Psi_A \tag{359}$$

where the † is assumed to operate on the quantum operators, not on the 4-vector components of Ψ, and the indices AB... designate the spinor components of the vector.
We would first have to show that there is Hamiltonian which produced these equations of motion. There is

$$H = -\int \left(\Psi(t,x)^T\gamma^0\gamma^k\partial_k\Psi(t,x) + \Psi(t,x)^T im\gamma^0\Psi(t,x)\right) d^3x \tag{360}$$

$$= -\int \left(\Psi_A \sum_C \gamma^{AC0}\gamma^{CBa}\partial_a\Psi_B + i\gamma^{AC0}m\Psi_C\right) \tag{361}$$

where ^{T} is the transpose of the 4-vector components of Ψ not the quantum operator aspect of Ψ. We also specify the communtation relations of Ψ to be

$$\Psi_A(t,x)\Psi_B(t,x') + \Psi(t,x')\Psi(t,x) = \delta_{AB}\delta(x,x') \tag{362}$$

where $\mathbf{I}$ is the identity matrix in the spinor components.
The equations of motion produced by the Hamiltonian and the these anti-commutation relations are

$$\partial_t\Psi(t,x) = i\,[H,\Psi(t,x)] \tag{363}$$

which is just the Dirac equation

$$i\gamma^a\partial_a\Psi + m\Psi = 0 \tag{364}$$

Let us select a complete set of solutions of the Dirac equation $\{\psi_j\}$, and assume we can divide these into a set and its complex conjugate $\{\psi_j,\psi_j^*\}$, such that

$$\sum_A \psi_{Ai}\psi_{Aj} = 0 \tag{365}$$

$$\sum_A \psi^*_{Ai}\psi_{A,j} = \delta_{ij} \tag{366}$$

Then define

$$\Psi(t,x) = \sum_i \left(\mathbf{c}_i\psi_i(t,x) + \mathbf{c}_i^\dagger\psi^*(t,x)\right) \quad (367)$$

where

$$\{\mathbf{c}_i, \mathbf{c}_j\} = 0 \quad (368)$$
$$\{\mathbf{c}_i, \mathbf{c}_j\} = \delta_{ij} \quad (369)$$

the usual anticommutation relations for Dirac operators.
It is clear that $\Psi^\dagger = \Psi$, and that

$$\{\Psi_A(t,x), \Psi_B(t,x)\} = \sum_i (\psi^*_{Ai}(t,x)\psi_{Bi}(t,x') + \psi_{Ai}(t,x)\psi(t,x')_{Bi}(t,x')) = \delta_{AB}\delta(x,x')(370)$$

because of the completeness and the assumed orthonormality of the modes. (These modes are eigenmodes of the Hermitian Hamiltonian operator, and are thus a complete orthonormal set).
We can write Ψ in an explicitly real form, by defining projection operators

$$H_i = \mathbf{c}_i + \mathbf{c}^\dagger{}_i \quad (371)$$
$$\tilde{H}_i = i(\mathbf{c}_i - \mathbf{c}^\dagger{}_i) \quad (372)$$

such that

$$\{H_i, H_j\} = \{\tilde{H}_i, \tilde{H}_j\} = 2\delta_{ij} \quad (373)$$
$$\{H_i, \tilde{H}_j\} = 0 \quad (374)$$

Then

$$\Psi(t,x) = \frac{1}{2}H_i\mathcal{R}\rceil\psi_i + \tilde{H}_i\mathcal{I}\Updownarrow\psi_i \quad (375)$$

There is a large degree of ambiguity in this definition. Let us assume that we define new operators

$$\mathbf{d}_i = \sum_j (A_{ij}\mathbf{c}_j + B_{ij}\mathbf{c}^\dagger{}_j) \quad (376)$$

then the condition that **d**s obey the same commutation relations as **c**s is

$$\{\mathbf{d}_i, \mathbf{d}_j\} \to \sum_k (A_{ik}B_{jk} + A_{jk}B_{ik}) = 0 \quad (377)$$
$$\{\mathbf{d}_i^\dagger, \mathbf{d}_j\} \to \sum_k (A^*_{ik}B_{jk} + A_{jk}B^*_{ij}) = \delta_{ij} \quad (378)$$