Physics 501-20

Amplifier model

Consider a linear amplifier. One would like it to act such that $\Phi_{out} = \alpha\Phi_{in}$ and $\Pi_{out} = \alpha\Pi_{in}$. Of course for a quantum system, this would violate the commutation relation, since

$$[\Phi_{out}(t,x), \Pi_{out}(t,x')] = i\delta(x - x') \tag{1}$$

but since $[\Phi_{in}(t,x), \Pi_{in}(t,x')] = i\delta(x-x')$, we instead would get $[\Phi_{out}(t,x), \Pi_{out}(t,x')] = i\alpha^2\delta(x - x')$. We could arrange that $\Phi_{out} = \alpha\Phi_{in}$ and $\Pi_{out} = \frac{1}{\alpha}\Pi_{in}$, but this is what is what is called a phase sensitive amplifier– in which one of the phases of the field (the Φ) is amplified but the conjugate is deamplified.

Going to simple oscillators, we can arrange for two oscillators $Q_1,\ P_1,\ Q_2,\ P_2$, with Annihilation operators such that A_1, A_2 such that

$$\tilde{A}_1 = \cosh(r)A_1 + \sinh(r)A_2^\dagger \tag{2}$$
$$\tilde{A}_2 = \cosh(r)A_2 + \sinh(r)A_1^\dagger \tag{3}$$

Where the ˜ refer to the output, while the plain are the input. Then, if we define a coherent state, for the first input, $A_1 |\psi\rangle = a |\psi\rangle$ while $A_2 |\psi\rangle = 0$, then

$$\langle\psi| \tilde{A}_1 |\psi\rangle = \cosh(r)a \tag{4}$$
$$\langle\psi| \tilde{A}_2 |\psi\rangle = \sinh(r)a^* \tag{5}$$

Thus, the expectation value of the output into the 1st channel is amplified by $\cosh(r) > 1$ while the output into the second channel may be an amplified version of the input or deamplified if $\sinh(r) > 1$ or not. Note that the expectation value of both the Hermitian operators $(\tilde{A}_1 + \tilde{A}_1^\dagger$ and $i(\tilde{A}_1 - \tilde{A}_1^\dagger$ are amplified by that same factor of $\cosh(r)$. If this can be arranged then one has a phase insensitive amplifier (ie, both the phases of the signal are amplified by the same amount). How can this be arranged?

Let us consider the following model free-Lagrangian.

$$\mathcal{L}_f = \frac{1}{2}\left[\int \left((\partial_t\phi(t,x))^2 - (\partial_x\phi(t,x))^2\right) dx - \int \left((\partial_t\psi(t,y))^2 - (\partial_y\psi(t,y))^2\right) dy + (\partial_t q)^2\right] \tag{6}$$

The second term has a minus sign, and would result in the possibility of an infinite negative energy. We will assume that this Lagrangian is a good approximation to the real Lagrangian, as long of the two fields ϕ and ψ are sufficiently small. At a certain amplitude or energy, we will assume that non-linearities ensure that the energy is eventually has a lower bound.

These are free, uncoupled fields and a single free particle modeled by q, Now, at $x = y = -\epsilon$ (where we will take ϵ to zero ultimately) we will couple the fields to the the free particle. Furthermore, let us assume that there is a dirichlet mirror at at $x = 0$ and $y = 0$ such that $\partial_x\phi(t,0) = \partial_y\psi(t,0) = 0$. The coupling will be such that

$$\mathcal{L}_I = (\lambda\phi(t,-\epsilon) + \mu\psi(t,-\epsilon))\,\partial_t q \tag{7}$$

The equations of motion of this system are

$$- + \lambda \partial_t q(t) \delta(x+\epsilon) = 0 \tag{8}$$

$$+ \mu \partial_t q(t) \delta(y+\epsilon) = 0 \tag{9}$$

$$-\partial_t^2 q - \partial_t(\lambda\phi(t,-\epsilon) + \mu\psi(t,y)) = 0 \tag{10}$$

$$\partial_x \phi(t,0) = 0 \tag{11}$$

$$\partial_y \psi(t,0) = 0 \tag{12}$$

where $= \partial_t^2\phi - \partial_x^2\psi$ and $= \partial_t^2\psi - \partial_y^2\psi$.

The retarded solution for ϕ is

$$\phi(t,x) = \phi_0(t-x) + \phi_0(t+x) + \frac{1}{2}\lambda(q(t-|x+\epsilon|) + q(t+x-\epsilon)) \tag{13}$$

$$\begin{aligned} \partial_x^2 q(t-|x+\epsilon|) &= \partial_x(-\partial_t q(t-|x+\epsilon|)\sigma(x+\epsilon)) && (14)\\ &= \partial_t^2 q(t-|x-\epsilon|)\sigma^2(x+\epsilon) - \partial_t q(t-|x+\epsilon|)2\delta(x+\epsilon) && (15) \end{aligned}$$

where $\sigma(\xi)$ is +1 if $\xi > 0$ and is -1 otherwise. And thus

$$(-\partial_t^2 + \partial_x^2)q(t-|x+\epsilon|) = -\partial_t q(t)\delta(x+\epsilon$$

The solution for the ψ field is

$$\psi(t,y) = \psi_0(t-y) + \psi_0(t+y) - \frac{1}{2}\mu\partial_t(q(t-|y+\epsilon|) + q(t+y-\epsilon)) \tag{16}$$

and the equation for q is

$$\begin{aligned} \partial_t^2 q(t) &+ \lambda\partial_t(\phi_0(t+\epsilon) + \phi_0(t-\epsilon) + \frac{1}{2}\lambda(2q(t) + q(t-2\epsilon)) && (17)\\ &+ \mu\partial_t(\psi_0(t+\epsilon) + \psi_0(t-\epsilon) - \frac{1}{2}\mu(2q(t) + q(t-2\epsilon)) = 0 && (18) \end{aligned}$$

We now take the limit as $\epsilon \to 0$. We have

$$\phi(t,x) = \phi_0(t-x) + \phi_0(t+x) + \lambda^2(q(t+x)) \tag{19}$$

$$\psi(t,x) = \psi_0(t-x) + \psi_0(t+x) - \lambda^2(q(t+x)) \tag{20}$$

$$\partial_t^2 q(t) + (\lambda^2 - \mu^2)\partial_t q(t) = -2(\lambda\partial_t\phi_0(t) + \mu\partial_t\psi_0(t)) \tag{21}$$

$$\tag{22}$$

Note that if $\mu^2 > \lambda^2$, the system is unstable, $q(t)$ will exponentially run away (and thus so will the output, the $t+x$dependent parts of $\psi(t+x)$ and $\phi(t+x)$, until the neglected non-linearities take over.

Going to the Fourier transform space where the input goes as $e^{-i\omega t}$, we have

$$-\omega^2 q_\omega - i\omega(\lambda^2 - \mu^2)q_\omega = -i\omega(\lambda\phi_{0\omega} + \mu\psi_{0\omega}) \tag{23}$$

$$\tag{24}$$

or

$$
\begin{aligned}
q_\omega &= \frac{\mu\phi_{0\omega} + \lambda\psi_{0\omega}}{-i\omega + (\lambda^2 - \mu^2)} && (25)\\
\phi_{\omega in} &= \phi_{0\omega} && (26)\\
\phi_{\omega out} &= (\phi_{0\omega} + \lambda(q_\omega)) && (27)\\
\psi_{\omega in} &= \psi_{0\omega} && (28)\\
\psi_{\omega out} &= \psi_{0\omega} - \mu q_\omega && (29)\\
& && (30)
\end{aligned}
$$

Given an incoming wavepacket in the ϕ channel, $\phi_0(t-x) = \int_{\omega>0} \alpha_\omega e^{i\omega(t-x)} d\omega$ with $\alpha_\omega = 0$ for $\omega < 0$, in the limit as $t \to -\infty$, so that $\int \phi_0(t+x) \approx 0 d\omega$ for all $x < 0$.

Given the full Lagrangian, the conjugate momentum for ϕ will be

$$\pi_\phi(t,x) = \partial_t\phi(t,x) - \lambda\partial_t q(t)\delta(x) \quad (31)$$

$$\pi_\psi(t,x) = -\partial_t\psi(t,x) - \mu\partial_t q(t)\delta(x) \quad (32)$$

$$p(t,x) = \partial_t q - \lambda\phi(t,0) - \mu\psi(t,0) \quad (33)$$

In the limit as $t \to -\infty$, the norm of the above mode will be

$$< \phi, \phi >= \frac{i}{2}\left(\phi_0(t-x)^*\partial_t\phi_0(t-x) - \partial_t\phi_0(t-x)^*\phi_0(t-x)\right) dx = \int \omega|\alpha_\omega|^2 d\omega > 0 \quad (34)$$

Similary for $\psi_0(t) = \int \beta_\omega e^{i\omega t} d\omega$ we get a similar expression, except, since $\pi_\psi = -\partial_t\psi$, the norm switches sign. Thus the $\omega > 0$ modes are negative norm for ψ channel, and will thus be associated with creation operators in this ψ input channel.

The modes in which q starts off non-zero and ϕ and ψ are zero will decay exponentially for q, and will produce a single outgoing mode in the ϕ and ψ channels. If the intial state is taken to occur for $t \to -\infty$, this mode will exponentially convert itself into outgoing modes which go off to $x \to \infty$. I will neglect this isolated mode.

As $t \to \infty$, the incoming wavepacket will convert itself to outgoing modes. For the outgoing modes

$$\phi(t,x) = \int \left(\alpha_\omega + \lambda\frac{\lambda\alpha_\omega - \mu\beta_\omega}{-i\omega + (\lambda^2 - \mu^2)}\right) e^{-i\omega(t+x)} d\omega \quad (35)$$

$$\psi(t,x) = \int \left(\beta_\omega - \mu\frac{\lambda\alpha - \mu\beta_\omega}{-i\omega + (\lambda^2 - \mu^2)}\right) e^{-i\omega(t+x)} d\omega \quad (36)$$

To quantize the system, again the quantum operators Φ, Ψ, Q obey exactly the same equation of motion. Writing in terms of the ingoing modes as $t \to \infty$, we have

$$\Phi_{in} = \int_{\omega>0} A_\omega \frac{e^{-i\omega(t-x)}}{\sqrt{2\pi\omega}} d\omega + HC \quad (37)$$

$$\Psi_{in} = \int_{\omega>0} B_\omega^\dagger \frac{e^{-i\omega(t-x)}}{\sqrt{2\pi\omega}} d\omega + HC \quad (38)$$

While in the limit as $t \to \infty$, we have the outgoing operators

$$\Phi_{out} = \int_{\omega>0} \tilde{A}_\omega \frac{e^{-i\omega(t+x)}}{\sqrt{2\pi\omega}} d\omega + HC \tag{39}$$

$$\Psi_{out} = \int_{\omega>0} \tilde{B}^\dagger_\omega \frac{e^{-i\omega(t+x)}}{\sqrt{2\pi\omega}} d\omega + HC \tag{40}$$

From the solution to the equations, we find

$$\begin{aligned}
\tilde{A}_\omega &= A_\omega - \lambda \frac{2\lambda A_\omega + 2\mu B^\dagger_\omega}{-i\omega + (\lambda^2 - \mu^2)} && (41)\\
&= A_\omega \frac{-i\omega - \lambda^2 - \mu^2}{-i\omega + \lambda^2 - \mu^2} + B^\dagger \frac{2\lambda\mu}{-i\omega + \lambda^2 - \mu^2} && (42)\\
\tilde{B}^\dagger &= B^\dagger + \mu \frac{2\lambda A_\omega + 2\mu B^\dagger_\omega}{-i\omega + (\lambda^2 - \mu^2)} && (43)\\
&= \frac{-i\omega + \lambda^2 + \mu^2}{-i\omega + \lambda^2 - \mu^2} B^\dagger_\omega + \frac{2\lambda\mu}{-i\omega + \lambda^2 - \mu^2} A_\omega && (44)
\end{aligned}$$

Ie, this system produces a squeezed two mode state, the two modes being in the two ϕ and ψ channels.

If the initial state is a vacuum state, the outgoing state is not.

$$\langle 0| \tilde{A}^\dagger_\omega \tilde{A}_{\omega'} |0\rangle = \langle 0| B_\omega B^\dagger_{\omega'} |0\rangle \left| \frac{\lambda\mu}{-i\omega + \lambda^2 - \mu^2} \right|^2 = \delta(\omega - \omega') \left| \frac{\lambda\mu}{-i\omega + \lambda^2 - \mu^2} \right|^2 \tag{45}$$

and similarly for $\langle 0| \tilde{B}^\dagger \tilde{B} |0\rangle$. There is a non-zero expecation value of the number of particles in the final state, even if the incoming state was the vacuum.

In this case the amplification factor from Φ_{in} to Φ_{out}

$$\begin{aligned}
\cosh(r_\omega) &= \left| \frac{-i\omega - \lambda^2 - \mu^2}{-i\omega + \lambda^2 - \mu^2} \right| && (46)\\
&= \sqrt{\frac{\omega^2 + (\lambda^2 + \mu^2)^2}{\omega^2 + (\lambda^2 - \mu^2)}} && (47)
\end{aligned}$$

Plotting the $10\log_{10}(\cosh(r_\omega)^2)$ (the power amplifiction) vs $\log(\omega)$, we find that for $\omega < \lambda^2 - \mu^2$, the amplification is relatively constant. For $\lambda^2 - \mu^2 < \omega < \lambda^2 + \mu^2$ the amplification drops at 6dB/octave, until at $\omega > \lambda^2 + \mu^2$ the amplification is essentially unity with a log of 0.

If, on the other hand the incoming signal was in the ψ channel, but the output is in the ϕ, the power amplification is $|sinh(r)|^2$ which looks similar the ϕ channel amplification except that it keeps dropping forever by 6dB/octave.

And this is a typical op-amp amplifier curve as taken from http://www.learningaboutelectronics.com/Article amp-specifications-full-power-bandwidth

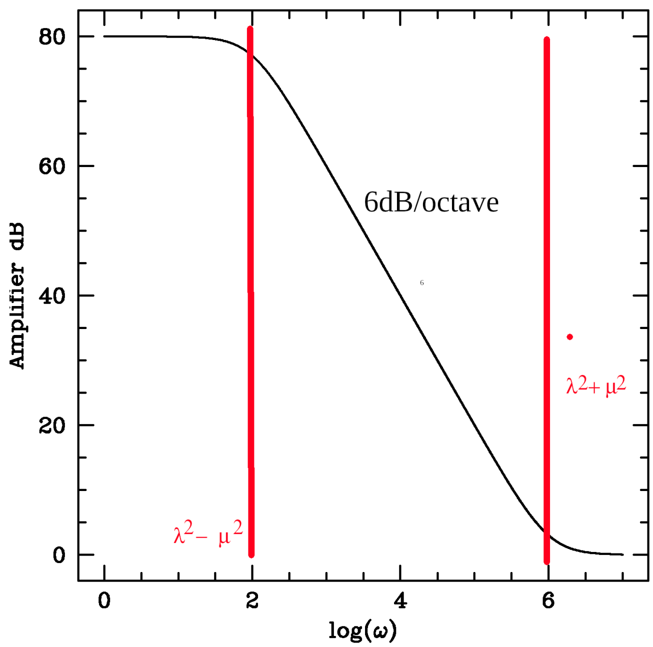

Figure 1: Amplifier with $\lambda^2 - \mu^2 = 1$ and $\lambda^2 + \mu^2 = 10000$

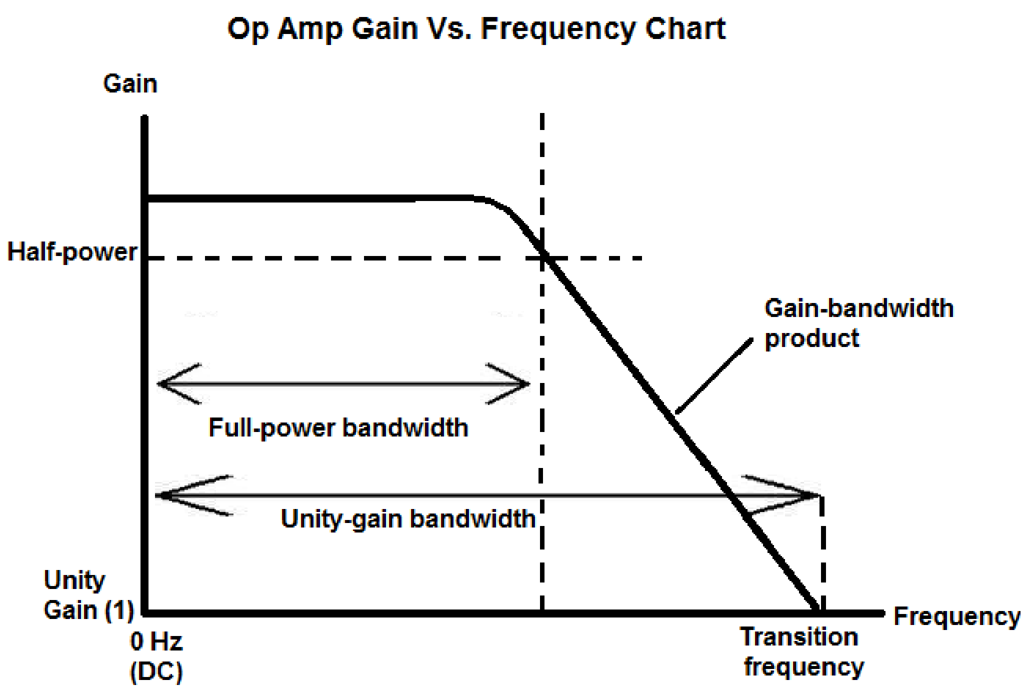

Figure 2: Typical Op-Amp gain vs frequency chart. Note that in this case it would be the equivalent sending the signal into channel Ψ and reading the output from channel Φ– ie the amplification is $\sinh(r_\omega)^2$