Physics 501-22

Weak Measurement

A weak measurement is one in which , while the measureing apparatus is correlated with the system attribute one is measuring, that correlation is not perfect. Ie, when one reads the measuring apparatus, one does not know exactly what the value of that attribute is.

One can model a measurement process by a simple model. Assume that one has a measuring apparatus with dynamic variables Q, Π where Π is the momentum conjugate to Q. $[Q,\Pi]=i$ To simplify the system, I will assume that that the Hamiltonian of the apparatus is 0 in the absence of interaction with the system.

If S is the attribute of the system one wants to measure, we will assume an interaction which occurs at time $t=0$ of the form

$$H_I = \epsilon\delta(t)\Pi S \tag{1}$$

In the Heisenberg representation, we have

$$\partial_t Q = \epsilon\delta(t)S(t) \tag{2}$$

$$\partial_t \Pi = 0 \tag{3}$$

with solution

$$Q(t) = Q_0 + \epsilon S(0)\Theta(t) \tag{4}$$

$$\Pi(t) = P_0 \tag{5}$$

where $\Theta(t)$ is the Heaviside step funtion which is 0 for t¡0 and 1 for t¿0. Ie, the interaction moves the pointer from its intial position to a final position which is displaced by S(0). This is of course an operator equation, and says nothing about the state of the system.

Let is assume that the state of the system is an eigenstate of $S(0)$ with eigenvalue s. If the initial state of the apparatus is $\phi_i(q)$, and the state of the system is $|s\rangle$, the s eigenstate of the S, we will have

$$\phi_s(t,q) = \phi_I(q)\Theta(-t) + \phi_I(q-\epsilon s)\Theta(t) \tag{6}$$

ie, if the "pointer" Q was concentrated around $Q=0$ before the interaction, it will be concentrated around ϵs after the interaction.

Because of the linearity of the Unitary interaction representing time evolution, if the apparatus was in an state independent of s before the interaction, with state $\psi(s)$, then after the interaction the state of the system will be

$$\Psi(s,q) = \sum_s \psi(s)\phi_I(q-\epsilon s) \tag{7}$$

If one now measures Q and finds the value q_0, thn the probability that one will obtain this value will be

$$\mathcal{P}(s,q_0) = |\psi(s)\phi_I(q_0-\epsilon s)|^2 \tag{8}$$

Unless the probability is highly peaked aroung $q_0 = \epsilon s_0$, one cannot associate the measured value of q_0 with some particular value of s from the measurement. The probability that s has some other value than s_0 could be relatively high. If ϕ_I is strongly peaked around 0, there may be large uncertainty in the value of s that q_0 corresponds to . At the same time, although one has approximately measured s, the state after the measurement still retains the phases and amplitudes of ψ. Ie, if ϕ_I is broad, encompassing a number values of s the state is not changed very much by that measurement.

The state of the system after the measurement of q_0 will be

$$\tilde{\psi}(s) = \psi(s)\phi_I(q_0 - \epsilon s) \tag{9}$$

Lets make the problem more definite. Lets assume that the intial state of the apparatus is a gaussian centered around 0

$$\phi_I(q) = \frac{e^{\frac{-q^2}{4\Delta^2}}}{\sqrt{\sqrt{2\pi}\Delta}} \tag{10}$$

Then

$$(\Delta q)^2 = \frac{1}{\sqrt{2\pi}\Delta}\int q^2 e^{\frac{-q^2}{2\Delta^2}}dq = \Delta^2 \tag{11}$$

and even if $\psi(s) = \delta_s$, the undertainty in the inferred value of s after a measurement of q_0, namely q_0/ϵ, will have the undertainty of $\frac{(\Delta s)^2\Delta^2}{\epsilon^2}$.

Weak Value

Let us now go back to the problem of two time conditions. Let us take a specific instance, namely S will be the "spin" in the directions θ

$$S = \cos(\theta)\sigma_3 + \sin\theta\sigma_1 \tag{12}$$

The Hamiltonian for the spin will be 0. The states $|+\rangle$ and $|-\rangle$ are the eigenstates of σ_1. The eigenstates of S with eigenvalues +1 an -1 will be designated by $|0\rangle$ and $|1\rangle$,with eigenstates

$$|1\rangle = \cos(\theta/2)|+\rangle + \sin(\theta/2)|-\rangle \tag{13}$$

$$|0\rangle = \cos(\theta/2)|-\rangle - \sin(\theta/2)|+\rangle \tag{14}$$

At 9AM σ_3 will be measured exactly, and its value wil be 1, so the intial state of the system will be $|1\rangle$. At 11AM, the spin in the μ directions

$$\sigma_\mu = \cos(\mu)\sigma_3 + \sin(\mu)\sigma_1 \tag{15}$$

is measured, and again, the value is +1. with eigenvector

$$|\uparrow\rangle = \cos(\mu/2)|+\rangle + \sin(\mu/2)|-\rangle \tag{16}$$

(17)

Useful in the following, we have

$$\langle + || 1 \rangle = \cos(\theta/2); \langle + || 0 \rangle = -\sin(\theta/2) \tag{18}$$

$$\langle 1 || \uparrow \rangle = \cos((\mu-\theta)/2); \langle 0 || \uparrow \rangle = -\sin((\mu-\theta)/2) \tag{19}$$

At 10 AM the operator S will be weakly measured. The question is what is the probability distribution of the inferred value of s from (is, of $\frac{q}{\epsilon}$) that weak measurement? Let us initially assume that the intial state of the measuring apparatus is again gaussian with an uncertainty in q of Δ. Thus the uncertainty in the measured value of s given by the measured value of q over Δ if the state for the system were in an eigenstate of S would be

$$\Delta s = \frac{\Delta}{\epsilon} \tag{20}$$

We will find if this uncertainty is much smaller than $s - s'$ (the difference between adjacent values of the eigenvalues) we will get one answer but if much larger, a very different answer.

We start with the intial state

$$\Psi_I(q) = \phi(q)|+\rangle = \phi(q)(\cos(\theta/2)|1\rangle - \sin(\theta/2)|0\rangle) \tag{21}$$

After the measurement interaction, we have

$$\Psi_\theta(q) = (\phi(q-\epsilon)\cos(\theta/2)|1\rangle - \phi(q+\epsilon)\sin(\theta/2)|0\rangle) \tag{22}$$

Recall the the procedure for finding probabilities given the final value are

$$\mathcal{P}(q) = \frac{|\langle(|q\rangle|f\rangle|^2}{\int |\langle(|q'\rangle|f\rangle|^2 dq'} \tag{23}$$

In our case we have

$$\mathfrak{p}(q) = |\left((\phi(q-\epsilon)\cos(\theta/2))^* \langle 1| - (\phi(q+\epsilon)\sin(\theta/2))^* \langle 0|\right)(|\uparrow\rangle)|^2 \tag{24}$$

Since ϕ and the θ dependent trig functions are all real, we can dispense with the complex congugation. Now, we have

$$\begin{aligned} \langle 1||\uparrow\rangle &= (\cos(\theta/2)\langle +| + \sin(\theta/2)\langle -|)(\cos(\mu/2)|+\rangle + \sin(\mu/2)|-\rangle) && (25)\\ &= \cos(\theta/2)\cos(\mu/2) + \sin(\theta/2)\sin(\mu/2) = \cos(\theta/2 - \mu/2) && (26)\\ \langle 0||\uparrow\rangle &= -\sin(theta/2 - \mu/2) && (27) \end{aligned}$$

so

$$\begin{aligned} \mathfrak{p}(q) &= (\phi(q-\epsilon)cos(\theta/2)cos((\mu-\theta)/2) - \phi(q+\epsilon)\sin(\theta/2)\sin((\mu-\theta)/2))^2 && (28)\\ \mathcal{P}(q) &= \frac{\mathfrak{p}(q)}{\int \mathfrak{p}(q')dq'} && (29) \end{aligned}$$

If Δ is small, $\phi(q-\epsilon)$ and $\phi(q+\epsilon)$ do not overlap, and one gets two probability peaks centered at $q = \pm\epsilon$. However if Δ is large, they overlap. We can expand ϕ as a function of ϵ to get

$$\phi(q \pm \epsilon) \approx \phi(q) \pm \epsilon \partial_q \phi(q) \tag{30}$$

so

$$\begin{aligned}
\mathfrak{p}(q) &\approx (cos(\theta/2)cos((\mu-\theta)/2) - sin(\theta/2)\sin((\mu-\theta)/2) & (31)\\
&\quad +(cos(\theta/2)cos((\mu-\theta)/2) + sin(\theta/2)\sin((\mu-\theta)/2))\epsilon\partial_q\phi(q) & (32)\\
&\approx cos(\theta/2)cos((\mu-\theta)/2) - sin(\theta/2)\sin((\mu-\theta)/2) & (33)\\
&\quad \times\phi\left(q + \frac{(cos(\theta/2)cos((\mu-\theta)/2) + sin(\theta/2)\sin((\mu-\theta)/2))}{(cos(\theta/2)cos((\mu-\theta)/2) - sin(\theta/2)\sin((\mu-\theta)/2))}\right) & (34)
\end{aligned}$$

Let us take $\mu = 2\theta$. Then we havethe argument be $\phi(q - \frac{\epsilon}{cos(\theta)}$. We note that The displacement of the exponential is larger than ϵ, which means that the measured value of S is larger than 1, which is the maximum eigenvalue of S.

If S has a more complex spectrum, one finds again as Δ becomes large, the displaement is given by

$$\delta q = \epsilon \frac{\langle \text{init}|S|\text{final}\rangle}{\langle \text{init}||\text{final}\rangle} \tag{35}$$

The factor in the interior is called the weak value for the S.

For the two level system, the uncertainty Δ must be larger than the separation of the eigenvalues times ϵ. For more complicated systems, for example if S were an N-equal level system) then Δ need only be larger than about $\sqrt{N}\epsilon$ times the eigenvalue separation for this to hold.

Just as in the Hardy chain, pobability that the final condition be satisfied goes to zero as the difference from the normal prejudice holds. In our example, the prefactor, $cos(\theta)^2$ goes to zero as $theta$ approaches $\pi/2$. When μ is π, then the final spin is in the opposite direction to the intial One is asking that the intial value of σ_3 be 1 and the final value of σ_3 be -1, which will rarely happen. However, as $\theta \to \pi/2$, we could make $\cos(\theta)$ appoach 200, giving a measurement of the spin (which is half of σ_3) to give the value 100. This amplification of the measurement may be useful in accurate measurement.

One can follow the above to generalize it. If S is the some operating in the system, we can write the solution as

$$\Psi = e^{-\frac{(q-\epsilon S)^2}{4\Delta^2}}|\text{init}\rangle \tag{36}$$

Expanding to first order in ϵ we have

$$\begin{aligned}
\mathfrak{p}(q) &\approx |\langle \text{final}|e^{-\frac{q^2}{4\Delta^2}}(1 - \frac{\epsilon S q}{2\Delta^2})|\text{init}\rangle|^2 & (37)\\
&= |\langle \text{final}||\text{init}\rangle e^{-\frac{q^2}{4\Delta^2}}(1 + \epsilon q \frac{\langle \text{final}|S|\text{initial}\rangle}{\langle \text{final}||\text{init}\rangle 2\Delta^2}) & (38)\\
&\approx |\langle \text{final}||\text{init}\rangle e^{-\frac{(q-\epsilon\frac{\langle \text{final}|S|\text{init}\rangle}{\langle \text{final}||\text{init}\rangle})^2}{4\Delta^2}}|^2 & (39)
\end{aligned}$$

The above has assumed that the function $\phi(q)$ is smooth– have a valid taylor expansion. The web page www.theory.physics.ubc.ca/weak-measure/weak-measure.html shows movies of the measurement of a spin 12.5 system (26 equally spaced levels for the spin in some direction) where the intial state $\phi(q)$ is taken to be a variety of different of functions. For functions which are not smooth (eg have discontinous derivatives) the probability distribution of Q retains many of the features for $\Delta \to 0$ even as Δ grows large. On the other hand, for smoother funtions (Gaussian, Lorentzian, etc) the distribution goes to one centered around 17 (for $\theta = \pi/4, \mu = \pi/2$) or $\sqrt{2}$ times 12.5. Note that one could regard this as a classical value. Regarding σ_3 as the z component and σ_1 as the x component, if both are maximum, that is like taking a vector and saying that its z compoenent and x component are both 12.5. Then the length of the vector is $\sqrt{12.5^2 + 12.5^2} = \sqrt{2}12.5 \approx 17$. Ie the weak value looks a lot like a classical value