Physics 501-22
Entanglement

Quantum Mechanics is a probabilistic theory. It gives probabilities of outcomes not values of outcomes. As with any probabilistic theory, it also contains the concept of correlation. If one has two separate things, two separate systems, then there is the concept of statistical independence and statistical correlation. In a non-statistical theory, things are always correlated. Given the intial conditions on the two system, the final results are deterministic and definite, and thus are always correlated. Two attributes $\mathcal{A}_1$ and $\mathcal{B}_2$ on systems 1 and 2, are correlated if $\{a_1\}$, and $\{b_2\}$ being the set of possible values of $\mathcal{A}_1$ and $\mathcal{B}_2$, and $P(a_1)$ and $P(a_2)$ are the probability of realising the specific values of a_1 and b_2 and $P(a_1, b_2)$ is the probability of realising both a_1, and b_2, then

$$P(a_1, b_2) \neq P(a_1)P(b_2) \tag{1}$$

for some value of a_1 and b_2. Ie, the probabliblity of a both is not simply equal to the probability of one times the probability of the other. If the equality holds for all a_1 and b_2 then the two are statistically independent and are said not to be correlated.

This concept applies to a quantum system as well. Two systems can be correlated if the probabilities of two outcomes are not the product of the probabilities of the individual outcomes. In quantum mechanics, if the state of the whole system is such that the probabilities of any two attributes, one from each system, obeys that inequality, then the states are called entangled.

Define the expectation value of some attribute $\mathcal{C}$ of some system. Then the expectation value of that attribute is equal to the probability of it having some value times that value, summed over all possible values for that attribute.

$$expectationvalue(c) = \sum_c cP(c) \tag{2}$$

It is also called the average value of c. In quantum mechanics, the attribute C is represented by some Hermitian operators C. If $|\psi\rangle$ is the quantum state of the system, then the expectation value of C is given by

$$< C >= \langle\psi| C |\psi\rangle \tag{3}$$

the possible values that the attribute $\mathcal{C}$ could take are the eigenvales of the operator C and the probability that that value is realised in the state ψ is $P(c) = |\langle\psi|c\rangle|^2$. where $|c\rangle$ is a unit vector which obeys

$$C |c\rangle = c |c\rangle \tag{4}$$

Thus on at least one level, entanglement is nothing mysterious, because correlations are a common feature of any probabilistic system. On the other hand, there are strange features of quantum system, which differentiate them from classical probabilistic system. This was first pointed out by Einstein, Podolsky and Rosen, amplified by Schroedinger, and sharped by Bell.

There are strange features of the quantum correlation, not shared by classical correlations. Consider a system which has two subsystems, with attributes $\mathcal{A}_\infty$ and $\mathcal{B}_\in$ and $\tilde{\mathcal{A}}_1$ and $\tilde{\mathcal{B}}_2$ of the two subsystems, which have operators A_1 , $\tilde{B}_2$ and B_2 associated with those attributes. Let as assume that the possible values for these attributes are all ± 1 (for example, the attributes attributes $\mathcal{A}_1$ could be whether or not the position was to the right or left of statue of John A Macdonald in Ottawa). Now consider a probability distribution such that the expectation value of all the attributes are 0 (for example that the probability of being to the left or right are equal.) but correlated (the expectation value of the $a_1 b_1$ is not zero.)

Let us look at

$$c = (a_1 b_2 + a_1 \tilde{b}_2 + \tilde{a}_1 b_2 - \tilde{a}_1 \tilde{b}_2) = (a_1(b_1 + b_2) + \tilde{a}_1(b_2 - \tilde{b}_2) \tag{5}$$

Since $b_2 + \tilde{b}_2$ has value ± 2 when $b_2 - \tilde{b}_2$ have value 0 and vice versa, Then c have value $\pm$ 2 or 0 and mean value of c^2 lies between 0 and 4.

Given quantum system on the other hand, we have

$$C = (A_1 B_2 + A_1 \tilde{B}_2 + \tilde{A}_1 B_2 - \tilde{A}_2 B_2) \tag{6}$$

and

$$C^2 = A_1^2 B_2^2 + A_1^2 \tilde{B}_2^2 + \tilde{A}_1^2 B_2^2 + \tilde{A}_1^2 \tilde{B}_2^2 + [A_1, \tilde{A}_1][B_2, \tilde{B}_2] \tag{7}$$

Since each of the operators has eigenvalues of ± 1 only, the square is the identity matrix. we have

$$C^2 = 4 + [A_1, \tilde{A}_1][B_2, \tilde{B}_2] \tag{8}$$

If the commutator of the As and Bs are zero, then we get exactly the same limits as in the classical case. However, if the commutators are imaginary and opposite (so that i times the commutators are Hermitian), the limits on C^2 can be larger than 4. The simplest situation is to use the Pauli sigma operators. Choosing $A_1 = \tilde{\sigma}_{x1}$ and $\tilde{A}_1 = \sigma_{y1}$ and $B_2 = \sigma_{y2}$ and$\tilde{B}_2 = \sigma_{x2}$, we get

$$C^2 = 4 + 4\sigma_{z1}\sigma_{z2} \tag{9}$$

To Maximize the expectation value of C^2 one needs a either the +1 eigenstate of σ_{z1} and of σ_{z2} or the -1 eigenstate of both, or any linear combination of them. Looking at C however, we find that the expectation value of $< C >$ for either of these is 0 (ie, an equal combination of the $\pm 2\sqrt{2}$ eigenstates of C) However both vectors $\frac{1}{\sqrt{2}}(|{+}{+}\rangle \pm |{-}{-}\rangle)$ are eigenstates of C with eigenvalues of $\pm 2\sqrt{2}$ The key point is that $B_2 + \tilde{B}_2$ does not have eigenstates of (0,$\pm$ 2) but rather has eigenstates $\sqrt{2}$. Furthermore, the two operators $B_2 \pm \tilde{B}_2$ do not commute and thus have no joint eigenstates. The values do not anticorrelate.

Ie, while the quantum correlations are similar tothe classical correlations, the quantum correlations, in this case are stronger than the classical correlation. Quantum correlations differ from classical correlations.

Note that for a quantum system, if A_1 and B_2 are operators on the two systems, if $|\psi\rangle = |\psi_1\rangle\,|\psi_2\rangle$, then

$$< A_1 B_1 >= \langle\psi_2|\,\langle\psi_1|\,A_1 B_2\,|\psi_1\rangle\,|\psi_2\rangle = \langle\psi_1|\,A_1\,|\psi_2\rangle\,\langle\psi_2|\,B_2\,|\psi_2\rangle =< A_1 >< B_2 > \quad (10)$$

Ie, the correlation function is zero for all operators $A_1,\ B_2$.These states are not entangled – there are no non-zero correlations between the two substates.

Note that whether or not a system is entangled can depend on how one divides the system into two parts. Let us consider a system made up on two harmonic oscillators, with dynamic variables X_1, P_1, X_2, P_2, and consider the Hamiltonian

$$H = \frac{1}{2}(P_1^2 + P_2^2 + X_1^2 + X_2^2 + 2\delta(t)X_1X_2) \quad (11)$$

The initial ground state of this system can be written in two different ways. Write new dynamic variable

$$Y_1 = (X_1 + X_2)/\sqrt{2} \quad (12)$$

$$Y_2 = (X_1 - X_2)/\sqrt{2}) \quad (13)$$

$$\Pi_1 = (P_2 + P_2)/\sqrt{2} \quad (14)$$

$$\Pi_2 = (P_1 - P_2)/sqrt2 \quad (15)$$

These obey $[Y_1, \Pi_1] = [Y_2, Pi_2] = i$ with all other commutators of 0. The Hamiltonian then is

$$H = \frac{1}{2}(\Pi_1^2 + \Pi_2^2 + (1 + \delta(t))Y_1^2 + (1 - delta(t))Y_2^2) \quad (16)$$

The initial (t¡0) ground state is

$$\psi(t) = e^{-it}e^{-\frac{y_1^2+y_2^2}{2}} = e^{-it}e^{-\frac{x_1^2+x_2^2}{2}} \quad (17)$$

Just after t=0, we have

$$\psi(t) = e^{-\frac{2y_1^2+0y_2^2}{2}} = e^{\frac{-x_1^2+x_2^2+2x_1x_2}{4}} \quad (18)$$

$$= e^{-x_1^2}e^{-x_2^2}e^{-x_1x_2} \quad (19)$$

While the first two terms are clearly a product of a function of x_1 times a function of x_2, the last term, equally clearly, is not. This state, while being a product of states for y_1 and y_2, it is not a product state for x_1 and x_2. This is a highly entangled state.

Ie, if we split the system into x_1 and x_2 the system's state after the interaction is an entangled state while if we spit it into the symmetric and antisymmetric parts, it is a product state and is not entangled. So, never let anyone talk about "an entangled state" without also telling you how the system is being split into a bipartite system.

Density Matrix

A bi-partite system, with some entangled state between the two, can be written as

$$|\psi\rangle = \sum_n q_n |\phi_{1n}\rangle |\phi_{2n}\rangle \tag{20}$$

where

$$\sum_{nm} q_n^* q_m \langle\phi_{1n}| |\phi_{1m}\rangle \langle\phi_{2,n}| |\phi_{2m}\rangle = 1 \tag{21}$$

so that the wave function is normalised.

Consider the operator operating only on the first Hilbert space, the Hilbert space of the first system

$$\rho_1 = \sum_{mn} [q_m^* q_n \langle\phi_{2m}| |\phi_{2n}\rangle] |\phi_{1n}\rangle \langle\phi_{1m}| \tag{22}$$

which is Hermitian. It is called a Density Matrix for the system 1 corresponding to the state $|\psi\rangle$ for the whole system.

Since the density matrix ρ_1 is Hermitian we can choose $|\phi_{1n}\rangle$ to be a complete orthonormal set of eigenvectors for ρ_1. Then

$$\rho_1 = \sum_n \lambda_n |\phi_{1n}\rangle \langle\phi_{1n}| \tag{23}$$

with $\lambda_n = |q_n|^2 \langle\phi_{2n}| |\phi_2 n\rangle$ and $\langle\phi_{2n}| |\phi_{2m}\rangle = 0$ for $n \neq m$. Ie, if we choose the set of orthonormal vectors which diagonalize ρ_1, then the set of vectors that $|\phi_{2n}\rangle$ must also be an orthonoral set, which diagonalize ρ_2. These are called the Schmidt decomposition of the pure state $|\psi\rangle$. The non-sero eigenvalues for ρ_2 are identical to those for ρ_1, and the sum of the eigenvalues must equal 1. Thus the eigenvalues of ρ_1 or ρ_2 can be interpreted as probabilities.

Decoherence

Given a subsystem of a larger system, any interaction between the subsystem and the rest of the total system will, in general create and entangled state between the two. Thus the subsystem instead of being decribed by a pure quantum state, will instead be described by a density matrix, if one ignores the rest of the system. This process of reducing a pure state to a mixed state, a state which is a sum of probabilities over pure states, is called decoherence. Decoherence is thus very easy to produce. Any kind of interaction between the system of interest and anything else, will produce decoherence.

Measurement

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What is a mesurement? This ultimately is one of the most contentious questions in quantum theory. But there is a simpler question. In carrying our a measurement one almost always inserts another system between the system one is interested in and oneself– a voltmeter, a microscope, ... One sets up

an interaction between the system one is interested in and the apparatus, such that making a measurement on the instrument gives on information about the system. Ie, this means that the apparatus starts on in some state, and after the interaction there is a correlation between the state of the apparatus and the state of the system one is interested in.

There is a very simple model one can use to study measuremnts, modeled on say an analog voltmeter. The voltmeter has a pointer and a dial. One one hooks it up to the system one is interested in, the pointer on the dial goes from some fiducial value of 0 to the voltage of interest. Lets choose our apparatus therefor to have dynamic degree of freedom which is the position of the pointer. Let us take this to Q. Assume that this has a conjugate momentum P_Q. How let us take the Hamiltonian of this apparatus to be 0. Ie, in the absense of interaction with the apparatus, it does not change.

Let us now hook up, for a time, the apparatus to the system. Lets say that we are measuring the position X of some particle. Lets take the Hamiltonian to be

$$H_I = \delta(t) P_Q X \tag{24}$$

The Heisenberg equation of motion for the apparatus is now

$$i\partial_t Q = [Q, H_I] = i\delta(t) X \tag{25}$$

$$\partial_t P = 0 \tag{26}$$

which gives

$$Q = Q_0 + X \tag{27}$$

$$P = P_0 \tag{28}$$

Ie, the interaction has displaced the pointer by a distance X. By measuring the position of the pointer after the interaction, one can determine what X was at the time of the measurement. Futhermore, one can also model the accuracy of the measuring apparatus. Let us assume that the measuring apparatus is set up so that the position of the pointer has a position with mean value of 0, and an uncertainty of Δ. Thus, initially we would have

$$< Q >=< Q_0 >= 0 \tag{29}$$

$$< Q^2 >= \Delta^2 \tag{30}$$

After the interaction, we would have

$$< Q >=< Q_0 > + < X > \tag{31}$$

$$< Q^2 > - < Q >^2 =< Q_0^2 > + < X^2) - < X >^2 + < Q_0 X > \tag{32}$$

If we assume that there is no correlation between Q_0 an X, the last term would be zero, and the variance in the measurmeent would equal the variance of the intial position of the pointer plus the intrinsic variance in X in its state just before the measurmemt.