QUANTUM NON-DEMOLITION

W. G. Unruh

Department of Physics, University of British Columbia, Vancouver, B.C., Canada.

1. THE QUANTUM LIMIT

As we have heard from Amaldi and Blair in this summer school, one must eventually worry about the quantum mechanical, rather than the strictly classical response of gravity wave detectors to an incoming pulse of gravitational radiation. This situation, where quantum effects begin to play a role in gravity wave detection, has come to be known as the quantum limit [1]. The purpose of these lectures will be to examine this quantum limit, to see in what sense it is actually a limit, and to discuss techniques for overcoming this limit to the detection of gravitational radiation.

To set the scene, let me begin by giving a rather loose derivation of the quantum limit. Consider a gravity wave detector idealised as in figure 1 as two masses coupled by a spring. In the absence of a gravity wave the equation of motion is given by

$$M\ddot{x} = -k(x - L)$$

where x is the separation between the masses, and L the "rest length" of the spring. The action of a gravity wave polarised parallel to the spring is effectively to change the effective length of the spring, x, to (1 + h)x. (See appendix A for a detailed treatment of this approach to the interaction of a gravity wave with an antenna). If we assume $x = L + \delta x$ where δx is small we have

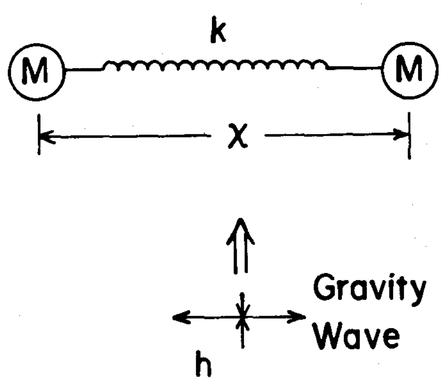

Fig. 1. Model resonant gravity wave detector.

$$M\delta\ddot{x} \simeq -k\delta x - khL \quad .$$

The term khL acts as a driving force. Taking h to be a pulse of width τ with $\tau \lesssim \sqrt{M/k}$, and amplitude h_o, we have

$$\delta x \simeq \delta x_o \cos(wt + \delta) + L\omega h_o \cos(\omega t + \delta')$$

where δx_o is the amplitude of oscillation in the absence of the gravity wave, δ, δ' are phase parameters depending on the initial conditions and time of arrival of the pulse and ω is $\sqrt{k/M}$. This gives the energy deposited in a stationary bar as

$$E \sim k(\omega L h_o \tau)^2$$

$$\sim kL^2(\omega\tau)^2 h_o{}^2 \cdot$$

If we tune the antenna so that $\omega\tau \sim 1$, we have maximum sensitivity and

$$E \sim kL^2 h_o{}^2 \sim M\omega^2 L^2 h_o{}^2 \cdot$$

taking $\omega \sim 10^4$ rad/sec, $M \sim 2 \times 10^3$ kg, and $L \sim 1$ metre as typical values for a bar type gravity wave antenna, we have

$$E \sim 10^{11} h_o{}^2 \quad \text{joules.}$$

The quantum limit is defined as that strength of gravity wave h_o which makes this energy equal to one quantum, ω. This limit is achieved when we have

$$h_o{}^2 \sim \hbar\omega/kL^2 \ .$$

For the parameters given above this gives $h_o \sim 10^{-20}$.

To convert this to a maximum distance to which such an antenna would be responsive, let us assume that $h \sim 1$ at a radius of one wavelength from the centre of the source [2]. For $\omega \sim 10^4$/sec, and assuming a $1/r$ fall off for h, we obtain

$$h_o \simeq c/\omega r_{max}$$

or

$$r_{max} \sim c/\omega h_o \sim 3.10^{21}\text{km} \sim 3.10^8 \text{ light years.}$$

Since few sources have an h of unity at the source, the maximum realistic radius would be reduced by at least a couple of orders of magnitude.

One of the more promising sources is the collapse of stars to form black holes [3]. Assuming the same rate for this as for supernova [4], one would have to be able to detect sources in the Virgo cluster of galaxies in order to expect a reasonable number of events per year. (Most experimentalists are unwilling to wait a lifetime for that one possible event). The distance to the Virgo cluster is greater than the above maximum "one quantum" distance. The question that must be answered is whether or not it is possible to detect a gravity wave whose strength is such as to deposit less than one quanta in a harmonic oscillator detector initially at rest.

Before continuing, I should point out that this quantum limit has nothing to do with any quantum properties of the gravity wave itself. Taking as a measure of the quantum properties of a field the number of quanta per cubic wave length, we find this to be [5]

$$N/\lambda^3 \simeq c^2\omega^2 h_o{}^2/Gh \sim 10^{30}$$

for $h_o \sim 10^{-20}$, far above any quantum limit. It is therefore the extremely small cross section of such antenna to gravitational radiation rather than any quantum properties of wave itself which force one to be concerned with quantum effects in the detector.

In dealing with such a resonant mass gravity wave detector, there are two possible points of view. In the first, one concentrates on the gravity wave itself, the oscillator acting simply as a transducer and amplifier of the gravity wave signal. Although this is in some sense the more fundamental point of view, it has been very profitable to concentrate one's analysis on the oscillator itself and assuming that the object of the experimental design is to detect the changes produced in the bar-oscillator by the gravity wave. As the danger inherent in this approach is that one can forget one's ultimate goal, that of detecting gravitational radiation and not of measuring properties of the oscillator, I will try to balance both points of view in these lectures.

Before beginning the description of recent works on the quantum limit in reference to gravity wave antenna, I would like to look at the classical papers, especially those by Heffner [6] and by Haus and Mullen [7] on the quantum noise limits for amplifiers and/or transducers. (There is basically no difference between an amplifier and a transducer except that in the latter case the output is of a different form from the input).

The argument advanced by Heffner was an uncertainty principle type argument, which suffers from the difficulties inherent in such arguments. He began with the uncertainty relation between the phase and quantum number of any linear system

$$\Delta n \; \Delta\phi \geq 1/2 \quad .$$

Strictly speaking no such uncertainty principle exists, and one can show that because of the discrete nature of the number operator spectrum, no conjugate Hermitian phase operator exists [8]. However, accepting the above relation, one can define a linear amplifier as one in which the output number and phase are related to the input number and phase by a relation of the form

$$n_o = Gn_i + \delta n$$

$$\phi_o = \phi_i + \delta\phi ,$$

where δn and $\delta\phi$ are assumed to be additive fluctuations introduced by the amplifier.

One now measures the output number and phase, and from the known amplification factor, G, one deduces information about the input number and phase. The output number and phase can only be measured to a certain accuracy Δn_o and $\Delta\phi_o$, with $\Delta n_o \Delta\phi_o \geq 1/2$. Let us assume that an optimal measurement has somehow been made on the output so as to make this an equality. If we assume the output number and phase uncertainties are uncorrelated with the fluctuations introduced by the amplifier, we can deduce the input number and phase with an accuracy given by

$$\Delta n_i^2 = (\Delta n_o^2 + \delta n^2)/G^2$$

$$\Delta\phi_i^2 = \Delta\phi_o^2 + \delta\phi^2 .$$

Now, we must have $\Delta n_i \Delta\phi_i \geq 1/2$, or else we will have measured the input to better than the quantum limit. Therefore we obtain

$$(\Delta n_o^2 + \delta n^2)(\Delta\phi_o^2 + \delta\phi^2)/G^2 \geq 1/4 ,$$

or

$$(1/4 + \Delta n_o^2 \delta\phi^2 + \delta n^2/4\Delta n_o^2 + \delta n^2 \delta\phi^2)/G^2 \geq 1/4 .$$

As the accuracy with which we measure the output number Δn_o is within the experimentalist's control, we can minimise the l.h.s. of the inequality by an appropriate choice of Δn_o. This gives

$$\Delta n_o^2 = |\delta n/2\delta\phi| .$$

We therefore must have

$$(1/4 + |\delta n \delta\phi| + |\delta n \delta\phi|^2)/G^2 \geq 1/4 .$$

As $|\delta n \delta \phi|$ is greater than 0, the l.h.s. is a monotonic function of $\delta n \delta \phi$. In order that this inequality be satisfied, $|\delta n \delta \phi|$ must be greater than the positive root of the equation

$$x^2 + x + (1 - G^2)/4 = 0$$

i.e.

$$\delta n \delta \phi \geq (G - 1)/2 . \qquad (1)$$

The amplifier must therefore add noise to the signal if the amplification G is greater than unity. In his paper, Heffner goes on to derive a minimum noise temperature for the amplifier under the hypothesis that the noise is additive Gaussian noise. The result he obtains is

$$T_{min} = \hbar\omega/(k\,ln((2G - 1)/(G - 1))) .$$

Although suggestive, the above analysis leaves a number of questions unanswered. Must the noise be additive noise, and could the noise not manifest itself, at least partially, as gain fluctuations rather than additive noise fluctuations? (After all, the phase transfer function is essentially just the argument of the complex gain, and phase uncertainty could therefore arise from gain fluctuations rather than from additive noise). Is the phase-number uncertainty valid for very weak signals where $n_i \sim 1$? What happens when there are many input channels, and in particular when the gain G becomes much less than unity (as happens in a gravity wave antenna where the conversion efficiency or gain of gravity waves to electrical signal is much less than unity). Re-examining eq. (1) in the case $G^2 < 1$, we find that $\delta n \delta \phi = 0$ is a perfectly acceptable solution. Is it true that a poor transducer can be perfectly noise free?

To answer these questions, a rather more rigorous analysis of an amplifier must be undertaken, and fortunately the job has already been done for us by Hans and Mullen [7]. My analysis will essentially follow theirs.

Let us define ψ and ϕ as two fields which are coupled linearly by the amplifier. We can define *in* fields Φ_i and Ψ_i as the fields which would be present in the absence of the amplifier coupling. Furthermore, Φ_o and Ψ_o are the free *out* fields which are present at the output from the amplifier. The linearity of the amplifier now implies that these *out* fields are *linear* functionals of the *in* fields

$$\Psi_o = \Psi_o(\Psi_i, \Phi_i)$$

$$\Phi_o = \Phi_o(\Psi_i, \Phi_i) \ .$$

For linear fields, we can expand Ψ_i and Φ_i in terms of the free field normal modes[9]. Regarding Ψ and Φ as quantum operators, the coefficients in the expansion will be creation and annihilation operator for these *in* modes of the field.

$$\Psi_i = a_\lambda \Psi_\lambda + a_\lambda^\dagger \ \Psi_\lambda^*$$

$$\Phi_i = b_\alpha \phi_\alpha + b_\alpha^\dagger \ \phi_\alpha^* \quad ,$$

where $\Psi_\lambda, \phi_\alpha$ are the free positive frequency modes of the field. Furthermore, the *out* fields, Ψ_o and Φ_o can also be expanded in normal modes to give the *out* creation and annihilation operators,

$$\Psi_o = \tilde{a}_\lambda \Psi_\lambda + \tilde{a}_\lambda^\dagger \ \Psi_\lambda^*$$

$$\Phi_o = \tilde{b}_\alpha \phi_\mu + \tilde{b}_\alpha^\dagger \ \phi_\mu^* \quad .$$

(I will reserve subscripts α, β, γ, δ for Φ and λ, μ, ν, ρ, for Ψ modes)

The linear relation between the *out* and *in* fields implies a linear relation between the *out* and *in* creation and annilhilation operators.

$$\tilde{a}_\lambda = M_{+\lambda\lambda'} a_{\lambda'} + M_{-\lambda\lambda'} \ a_{\lambda'}^\dagger \ + M_{+\lambda\alpha'} b_{\alpha'} + M_{-\lambda\alpha'} \ b_{\alpha'}^\dagger$$

$$\tilde{b}_\alpha = M_{+\alpha\alpha'} \ b_{\alpha'} + M_{-\alpha\alpha'} b_{\alpha'}^\dagger \ + M_{+\alpha\lambda'} a_{\lambda'} + M_{-\alpha\lambda'} \ a_{\lambda'}^\dagger \quad ,$$

where the M's are constant matrix elements, and the summation convention has been used.

The above M's are not arbitrary because both the *in* and *out* annihilation and creation operators must obey appropriate commutation relations [10]. Defining the matrix

$$M = \begin{bmatrix} (M_{+\alpha\alpha'}) & (M_{+\alpha\lambda'}) & (M_{-\alpha\alpha'}) & (M_{-\alpha\lambda'}) \\ (M_{+\lambda\alpha'}) & (M_{+\lambda\lambda'}) & (M_{-\lambda\alpha'}) & (M_{-\lambda\lambda'}) \\ (M^*_{-\alpha\alpha'}) & (M^*_{-\alpha\lambda'}) & (M^*_{+\alpha\alpha'}) & (M^*_{+\alpha\lambda'}) \\ (M^*_{-\lambda\alpha'}) & (M^*_{-\lambda\lambda'}) & (M^*_{+\lambda\alpha'}) & (M^*_{+\lambda\lambda'}) \end{bmatrix}$$

and the two column vectors

$$A = \begin{Bmatrix} (b_\alpha) \\ (a_\lambda) \\ (b^\dagger_\alpha) \\ (a^\dagger_\lambda) \end{Bmatrix} \qquad \tilde{A} = \begin{Bmatrix} (\tilde{b}_\alpha) \\ (\tilde{a}_\lambda) \\ (\tilde{b}^\dagger_\alpha) \\ (\tilde{a}^\dagger_\lambda) \end{Bmatrix}$$

we can write the linear transformation produced by the amplifier as

$$\tilde{A} = MA .$$

Define the matrix P by

$$P = A\,A^t - (A\,A^t)^t ,$$

where the superscript t means the transpose of the preceeding matrix while maintaining the order of the quantum operators in any one of the matrix elements. The commutation relations for the *in* operators imply

$$P = \begin{Bmatrix} 0 & -I \\ I & 0 \end{Bmatrix}$$

where I is the identity matrix. Defining $\tilde{P}$ in the same way from the *out* operators, we have

$$\tilde{P} = P .$$

From the relation between *in* and *out* operators we have

$$\tilde{P} = P = AA^t - (AA^t)^t = MA\,A^tM^t - M(A\,A^t)^tM^t = M\,P\,M^t .$$

The matrix M must therefore preserve the form of the matrix P (i.e. be unitary with respect to P.)

This analysis is very familiar from S-matrix scattering theory [10]. The amplifier or transducer in this case provides the coupling between the various fields being measured and the output measuring fields. The unitarity condition on M is just the unitarity condition on the scattering matrix [11]. Furthermore, since we are here examining linear interactions, the M matrix coefficients correspond to the Bugoliubov coefficients in linear scattering theory.

In the following I will assume that the Φ field is the one which we are attempting to measure, while the Ψ field represents a field which we are able to measure readily. (I will not analyse how we measure the Ψ field [12]). In particular I will assume that we are monitoring only certain specific Ψ-modes which I will designate with subscripts $\tilde{\mu},\tilde{\nu}$, and that this monitoring consists of measuring the number of quanta in these modes. The number operator for particles in any one of these modes is given by

$$N_{\tilde{\mu}} = \tilde{a}^{\dagger}_{\tilde{\mu}}\, \tilde{a}_{\tilde{\mu}}.$$

Now, let the state of the system be designated by $|p\rangle$ and let $|0\rangle$ be the *in* vacuum state. $|0\rangle$ is thus the state in which initially there are no quanta of any type present so that

$$a_{\mu}\, |0\rangle = b_{\alpha}\, |0\rangle = 0: \quad \forall \alpha,\mu.$$

The expectation value of $N_{\tilde{\mu}}$ can now be written as

$$\langle p|\ \tilde{N}_{\tilde{\mu}}\ |p\rangle = \langle p|\ :\tilde{N}_{\tilde{\mu}}:\ |p\rangle + \langle 0|\ \tilde{N}_{\tilde{\mu}}\ |0\rangle,$$

where :N: indicates that N is normal ordered with respect to the *in* annihilation and creation operators. The first term is thus zero when $|p\rangle$ is the vacuum state. The second term, on the other hand, is always present, and thus represents a noise term which is independent of the input to the system. This noise term is, furthermore, obviously additive.

Writing $\tilde{N}_{\tilde{\mu}}$ in terms of the *in* operators we obtain

$$\begin{aligned}\langle 0|\tilde{N}_{\tilde{\mu}}|0\rangle &= \langle 0|\,(M^{*}_{+\tilde{\mu}\lambda}\, a^{\dagger}_{\lambda} + M^{*}_{+\tilde{\mu}\alpha} b^{\dagger}_{\alpha} + M^{*}_{-\tilde{\mu}\lambda}\, a_{\lambda} + M^{*}_{-\tilde{\mu}\alpha} b_{\alpha}) \\ &\quad \times\ (M_{+\tilde{\mu}\nu} a_{\nu} + M_{+\tilde{\mu}\beta} b_{\beta} + M_{-\tilde{\mu}\nu} a^{\dagger}_{\nu} + M_{-\tilde{\mu}\beta} b^{\dagger}_{\beta})\,|0\rangle \\ &= \sum_{\lambda} |M_{-\tilde{\mu}\lambda}|^2 + \sum_{\alpha} |M_{-\tilde{\mu}\alpha}|^2 ,\end{aligned}$$

where I have used the summation convention over λ, α, ν, β in the second expression.

To relate this to the gain of the system when there are particles in the input modes, let us choose $|p\rangle$ to be a state with n particles in a particular state $\phi_{\gamma'}$. We therefore define $|p\rangle$ by

$$b^{\dagger}_{\gamma'}\, b_{\gamma'}\, |p\rangle = n\,|p\rangle$$

$$a_{\gamma}\, |p\rangle = b_{\alpha}|p\rangle = 0, \quad \forall \lambda,\alpha \neq \gamma'$$

The first term, $<p| : \tilde{N}_{\tilde{\mu}} :|p>$ can now be calculated. We obtain

$$<p| : \tilde{N}_{\tilde{\mu}} : |p> = (|M_{+\tilde{\mu}\gamma'}|^2 + |M_{-\tilde{\mu}\gamma'}|^2)n.$$

We can therefore define the amplifying factor for the γ' mode to be converted to a $\tilde{\mu}$ mode as

$$G_{\tilde{\mu}\gamma'} = |M_{+\tilde{\mu}\gamma}|^2 + |M_{-\tilde{\mu}\gamma'}|^2 \quad .$$

Similarly we find the amplifying factor, $G_{\tilde{\mu}\lambda'}$, for the $\psi_{\lambda'}$ *in* mode to be converted to a $\tilde{\mu}$ mode as

$$G_{\tilde{\mu}\lambda'} = |M_{+\tilde{\mu}\lambda'}|^2 + |M_{-\tilde{\mu}\lambda'}|^2 \quad .$$

The total amplifying factor $G_{\tilde{\mu}}$ can now be defined as

$$G_{\tilde{\mu}} = \sum_{\lambda'} G_{\tilde{\mu}\lambda'} + \sum_{\gamma'} G_{\tilde{\mu}\gamma'} .$$

$G_{\tilde{\mu}}$ therefore represents the number of particles in the $\tilde{\mu}$ *out* mode if there is one particle in every one of the possible *in* modes.

However, from the unitarity condition on the M matrix we have

$$\delta_{\tilde{\mu}\tilde{\nu}} = (\sum_{\lambda} M^*_{+\tilde{\mu}\lambda} M_{+\nu\lambda} + \sum_{\alpha} M^*_{+\tilde{\mu}\alpha} M_{+\tilde{\nu}\alpha})$$

$$- (\sum_{\lambda} M^*_{-\tilde{\mu}\lambda} M_{-\nu\lambda} + \sum_{\alpha} M^*_{-\tilde{\mu}\alpha} M_{-\tilde{\nu}\alpha}) \; ,$$

which gives

$$1 = (\sum_{\lambda} | M_{+\tilde{\mu}\lambda}|^2 + \sum_{\alpha}|M_{+\tilde{\mu}\alpha}|^2)$$

$$- (\sum_{\lambda} |M_{-\tilde{\mu}\lambda} |^2 + \sum_{\alpha}|M_{-\tilde{\mu}\alpha}|^2)$$

$$= G_{\tilde{\mu}} - 2 < 0 \mid \tilde{N}_{\tilde{\mu}} \mid 0 > \quad .$$

This gives us a relation between the noise and the gain:

$$<0|\tilde{N}_{\tilde{\mu}} |0> = (G_{\tilde{\mu}} - 1)/2 \quad .$$

This is precisely the Heffner result. This derivation, however, makes it clearer what the amplifying factor $G_{\tilde{\mu}}$ means. It is not the amplifying factor for any one possible input mode, but the sum of the amplifying factors for all possible input modes, including the modes we are measuring at the output. It is also clear that $G_{\tilde{\mu}}$ can never be less than unity. Except in the trivial transducer case, in which $G_{\tilde{\mu}}$ is unity and the amplifier at best converts one quanta into one quanta of a different type, the noise will always be non zero.

This also suggests the procedure which must be followed to minimise the noise which is to make $G_{\tilde{\mu}\alpha}$ and $G_{\tilde{\mu}\lambda}$ as small as possible for all α or λ modes except the one of interest. The amplifier should be as insensitive to all other modes as possible. An ideal situation, for example, would be to design the amplifier so that

$$M_{-\tilde{\mu}\alpha} = M_{+\tilde{\mu}\alpha} = 0$$

except for the particular mode α' one wished to measure, and to have

$$|M_{-\tilde{\mu}\alpha'}| = |M_{+\tilde{\mu}\alpha'}|$$

and finally to have

$$M_{+\tilde{\mu}\tilde{\mu}} = 1 ,$$

with all other components being zero. In general, for gravity wave detection, the amplifying factor for gravity wave modes is extremely small (of the order of 10^{-30} or less for the usual bar type detector) giving an extremely small limit to the quantum noise due to the amplification of the gravity wave. This demonstrates that in principle at least the so called quantum limit is in fact not a limit to gravity wave detection. The quantum limit for the usual detector arises because one has not minimised the non essential amplifying factors; one has allowed the bar to amplify not only the gravity waves but also other non essential modes.

In order to present a slightly more physical picture of what is happening, we notice that $G_{\tilde{\mu}}$ can be greater than unity only if some of the M_- matrix elements are non zero. These matrix elements represent the transformation of *in* annihilation operators into *out* creation operators. Since annihilation operators are associated with positive frequency modes, while creation operators are associated with negative frequency modes, the linear transformation from the *in* to the *out* modes must be time dependent. We have

$$\Psi_o(t,x) = \int (M_1(t,t',x,x')\ \Psi_i\ (t',x') + M_2(t,t',x,x')\ \Phi_i(t',x'))dt'dx' ,$$

where the transformation matrices M_1 and M_2 (corresponding to $M_{\pm\mu\lambda}$ and $M_{\pm\mu\alpha}$) must be time dependent. The amplifier must therefore supply a time dependent coupling of the fields to each other if amplification, rather than simply transduction, is to take place.

This allows us to give a simple picture of the physical cause of the quantum noise in any amplifier. Since time dependence in nature is the result of dynamic processes, the time dependence introduced by the amplifier must be due to some dynamic variables. In treating them as classical functions in the interaction produced by the amplifier

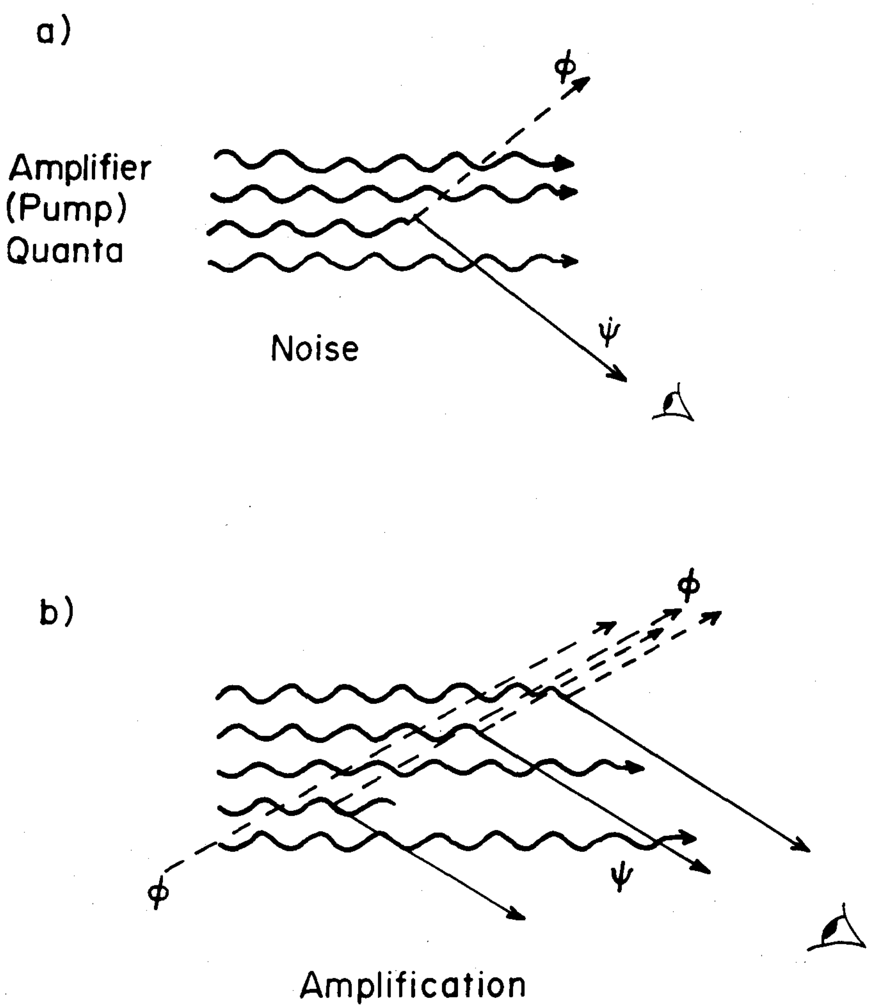

Fig. 2. Symbolic relation between amplification and noise in a detector.

rather than as quantum operators with quantum fluctuations, these dynamic variables must be highly excited, corresponding naively to a system with an extremely large number of quanta. Each of these quanta couple to the Φ and Ψ states, and can thus decay into a Φ plus a Ψ quanta. This decay has a certain natural rate even in the absence of any Ψ or Φ quantum. It is this natural decay of these amplifier quanta which gives the additive noise (see figure 2a). In the presence of Φ quanta say, this decay process is stimulated, and, the more Φ quanta present, the more rapidly the decay proceeds, emitting Ψ quanta at a rate proportional to the number of Φ quanta present (fig. 2b). This is the amplification process. However, as first pointed out by Einstein [13], the stimulated rate and the spontaneous rate are directly proportional to each other. This is what leads to the intimate relation between the amplification and noise of any amplifier.

The above review of the classical papers therefore leads us to the following conclusions.

1. *The amplifier should be designed to couple only the input modes of interest to the relevant output of the amplifier. Any additional couplings will increase the noise without increasing the sensitivity.*

2. *The ultimate quantum limit is set by the fact that the amplifier can act as a source of gravitational radiation. Because there is no way of telling whether the output of the amplifier was due to the reception and amplification of a gravity wave pulse, or due to the decay of one of the amplifier quanta into a gravity wave plus an output quanta, this process will produce an inescapable noise in the output. However, this source of noise is about 30 orders of magnitude below the naive "quantum limit": and can be disregarded for the present.*

The problem now arises as to how we can design the detector of gravity waves, or more specifically, whether and how we can couple to a harmonic oscillator type gravity wave detector in order to minimise the coupling of all extraneous inputs to the detector to the output (and particularly minimise the coupling of the readout system itself to the output of the detector). This will be the problem which I will address next.

Before proceeding with the criterion for the development of a detector which will evade the naive quantum limit, it may be instructive to examine a semi-realistic detector model so as to identify the various rather abstract components which I have discussed above. The model is that of a laser interferometric readout of a harmonic oscillator type gravity wave detector. I will not analyse this system completely, but rather point out the essential features. In appendix B a simple harmonic oscillator transducer (with G = 1) is analysed in detail.

Figure 3 gives the essential parts of such a readout system. Any motion of the

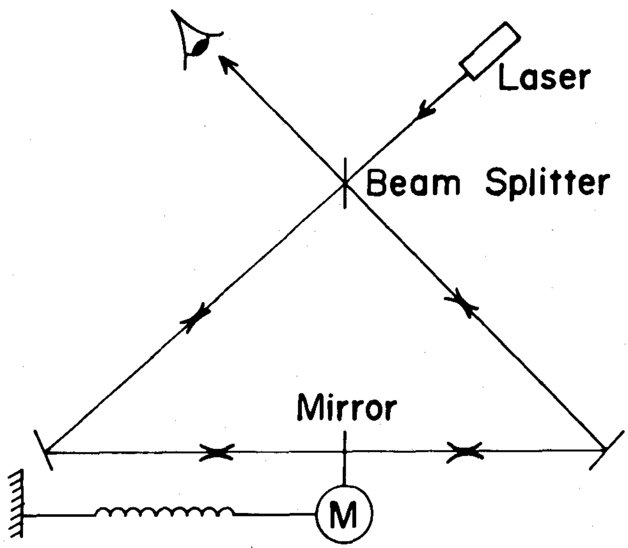

Fig 3. Laser interferometer readout of gravity wave detector.

oscillator will alter the phase relation of the beams at the beam splitter, increasing or decreasing the number of photons measured at the other side of the beam splitter. The system is assumed carefully adjusted so that when the oscillator is in its equilibrium position, all of the returning photons are transmitted back toward the laser by the beam-splitter giving no signal at the output.

The incoming gravity wave is the field Φ which we wish to measure. The interaction with the oscillator is linear (the "force" on the oscillator is just proportional to the amplitude of the gravity wave). The interaction of the light wave is via the mirror. We can model the mirror by a potential V centred at the position, q, of the oscillator. Let us assume that the incident light is in a coherent state $\xi(t,x)$, where we choose $\xi(t,x)$ to solve the electromagnetic field equations with the mirror at the equilibrium position of the oscillator.

Now the electromagnetic field E(t,x) will obey a field equation of the symbolic form

$$\Box E = V(x - q)\ E\ .$$

I assume that ξ obeys

$$\Box\xi = V(x)\xi \ .$$

Define the readout field, Ψ, by

$$\Psi = E-\xi$$

which obeys

$$\Box\Psi = [V(x-q) - V(x)]\xi + V(x-q)\Psi \ .$$

If the oscillator is disturbed only very little from its equilibrium position, we can linearise the above equation

$$\Box\Psi = V'(x)\ q\ \xi + V(x)\Psi \ .$$

This will hold only so long as the expectation value of Ψ remains much less than the magnitude of ξ, (which requires, for example, that the amplitude of oscillation must be much less than a wavelength). However, if this linearisation is valid, then Ψ will depend linearly on q which depends linearly on the gravity wave.

Initially the field Ψ will be in its vacuum state. Any motion of the mirror will create Ψ quanta. At the same time, the radiation pressure proportional to $\xi\Psi$ will produce a back reaction on the oscillator. The system is designed so that the ξ signal is not transmitted along the readout path to the eye. However, some of the Ψ modes, being of a different frequency than the ξ modes (due to the doppler shift produced by the moving mirror) and having different phase relations in the two paths of the split beam, will be transmitted to the eye and act as the readout signal.

This system is far from ideal. A detailed analysis which will be presented elsewhere [14], shows that there is a large noise component due to the coupling of the readout field Ψ to the oscillator. Essentially this noise can be regarded as due to the light quanta in the split beam exerting random uncorrelated δ-function type forces on the two sides of the mirror, and exciting the oscillator, which then produces quanta in the Ψ readout modes.

This system acts not only as a transducer, but also as an amplifier, with the necessary time dependence being supplied by the classical coherent light source from the laser.

II OPTIMAL QUANTUM DETECTION

The results of the last section have suggested that for a linear detection scheme, one should design the detector system in such a way that the output to the readout system should be independent of the readout system itself. It did not, however, supply any suggestions as to how this could be accomplished.

Let us examine the problem from a different point of view first suggested by Hollenhorst [15]. Although he uses the language of quantum decision theory [16] , the results of interest can be more easily studied using only ideas from elementary quantum mechanics.

The object of his analysis is to describe the limits imposed by quantum mechanics on the measurement of the changes produced by a gravity wave on the state of the oscillator. The gravity wave itself is assumed to be a classical force in that all quantum fluctuations of the gravitational field are ignored, as is the possibility that the detector could generate gravitational radiation. Furthermore, the oscillator is assumed to be free of interaction with anything else.

The oscillator is assumed to be in a known initiated state, $|i\rangle$. In the absence of any interactions it will remain in this state. The effect of the classical force on the oscillator will be to change this state $|i\rangle$ to some different state $|f\rangle$. One now wishes to determine either what that final state $|f\rangle$ is or to determine whether or not any change has taken place. Because these two states, $|f\rangle$ and $|i\rangle$, are in general non-orthogonal, finding optimal answers to these two possible questions will be incompatible. In particular, the optimal techniques for determining whether or not some interaction has taken place is given by determining whether the system is still in the state $|i\rangle$ after the action of the classical force. If one finds it is not in the state $|i\rangle$, one knows for certain that something has altered the state, and that $|f\rangle$ is not identical with $|i\rangle$. However, a determination that the system is still in the state $|i\rangle$ does not allow the conclusion that $|f\rangle$ and $|i\rangle$ are identical (i.e. that there has been no force acting on the oscillator). In particular, there is a probability [17]

$$P = |\langle f|i\rangle|^2$$

that the system will still be found to be in the state $|i\rangle$ even if $|f\rangle$ and $|i\rangle$ differ. If P is sufficiently large for some choice of initial state $|i\rangle$ and for some amplitude for the gravity wave, then the probability of detecting that a change has been produced in the state of the system becomes small, and that particular gravity wave becomes undetectable.

This false-null probability, P, depends both on the initial state $|i\rangle$ of the oscillator and on the effect the gravity wave has on the oscillator. For a classical force, the effect is easily calculated to be

$$|f\rangle = \exp\,(i(\alpha q + \beta p/\Omega))\; i\rangle \;,$$

where p, q are the canonical momentum and position operators for the oscillator with Hamiltonian

$$H = (p^2 + \Omega^2 q^2)/2 \;,$$

and α and β are the cos and sin Fourier components of the force at frequency Ω. One can readily calculate P for any given initial state $|i\rangle$,

$$P = |\langle i|\; \exp i\; (\Omega\alpha q + \beta p)\quad |i\rangle|^2 \quad .$$

Hollenhorst has calculated this probability for various possible initial states. For $|i\rangle$ the ground state of the oscillator, one obtains

$$P_{ground} = e^{-(\alpha^2 + \beta^2)/2\,\Omega} .$$

In order that the change produced by the gravity wave be detectable, P must be small, from which we obtain

$$\alpha^2 + \beta^2 \gtrsim 2\Omega .$$

The classical energy deposited by the force in a bar initially at rest is given by

$$E_r = (\alpha^2 + \beta^2)/2,$$

from which we obtain exactly the "quantum limit".

$$E_r \gtrsim \Omega .$$

Hollenhorst furthermore shows that any coherent state [18] gives precisely the same result.

Since the result depends on the initial state chosen, this result can be changed. In particular, he calculates the probability P for the energy eigenstates. He finds that P decreases roughly as 1/n for any given amplitude $(\alpha^2+\beta^2)^{\frac{1}{2}}$ of the wave where n is the number of quanta in the initial state $|i\rangle$ of the oscillator. Figure 4 is adapted from Hollenhorst to illustrate the dependence of P on n. There P is plotted versus $\alpha^2+\beta^2$ for n=0 and n=10. This illustrates explicitly that there is nothing about the quantum nature of the oscillator itself which limits the sensitivity of the detector. One does, however, have to choose the initial state of the oscillator carefully.

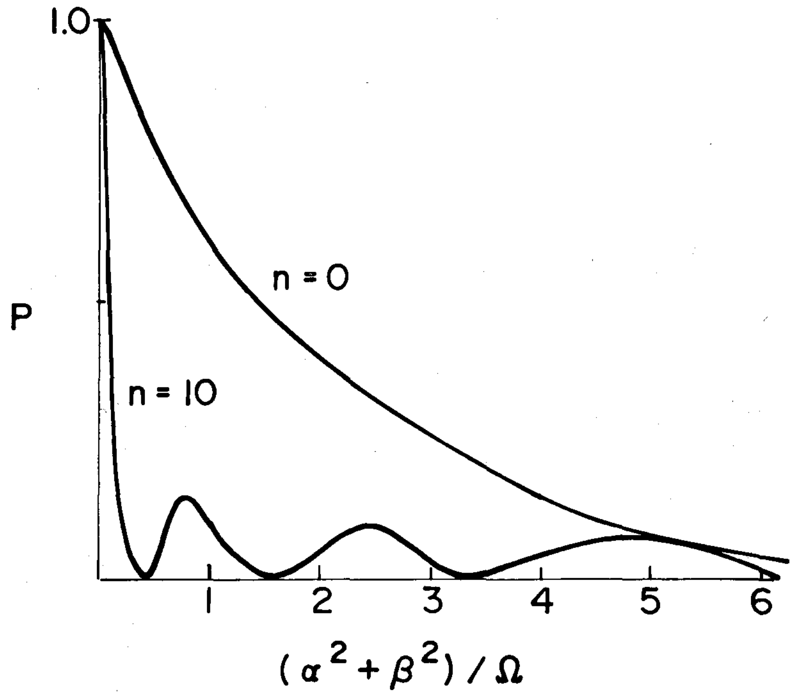

Fig. 4. Probability of non detection vs. signal strength for oscillator in ground state and in 10 quantum state for energy detection coupling.

As a final example, he also calculates P for a set of states he calls wave packet states. These are states which have a minimum uncertainty in that $\Delta p \Delta q = \frac{1}{2}$, but in which the wave packet is squeezed in one direction and expanded in the other direction in comparison with the vacuum state. The simplest such state is obtained by applying the operator

$$S(\sigma) = \exp - [i\sigma(p^2 + \Omega^2 q^2)/2\Omega]$$

to the ground state, $|0\rangle$, to give the initial state

$$|i\rangle = S(\sigma)\ |0\rangle \ .$$

In this case one finds the condition

$$e^{\sigma}\ \alpha^2 + e^{-\sigma}\ \beta^2 \gtrsim 2 \ .$$

For the correct phase of the force (i.e. $\beta^2 \ll \alpha^2$) this gives a much improved sensitivity over the ground state as the initial state.

Although Hollenhorst's analysis is helpful, it still leaves some questions unanswered. In particular, how does the measuring process itself affect the above analysis; how does one determine $|i\rangle$; and how does one determine whether or not the system is still in the state $|i\rangle$ after a time. Essentially, his analysis leaves out an analysis of the readout system on the oscillator, and the effect of this readout on the state of the oscillator.

The readout system must be coupled to the oscillator. The system must be designed so that the state of the readout depends on the state of the oscillator. However, the quantum nature of the readout system means that the coupling must be sufficiently strong so that the different effects on the readout caused by different states of the oscillator can be reliably distinguished. (The readout system suffers from the same difficulties as the oscillator itself in that weak effects can lead to a high probability of no detectable change in the readout state). A strong coupling of the oscillator to the readout implies a strong reverse coupling as well, implying that the state of the oscillator will also depend on the state of the readout system.

The quantum nature of the measurement process leads therefore to two difficulties. The first is that the change produced by the gravity wave may be too small to be detected reliably, while the second is that the readout system can affect the state of the oscillator itself leading to possible noise.

The Hollenhorst analysis offers a possible way out of this dilemma [19]. In particular, the optimal strategy according to him is to measure at later times the projection operator $|i\rangle\langle i|$. This operator is time indepedent in the Heisenberg representation in the absence of any interaction with the gravity wave. Any change in this operator must therefore occur because of some outside influence. This suggests that the conclusion one should draw is that for optimal detection of the influence of a gravity wave on the detector one should "measure" an operator Z which is time independent in the absence of a signal, but which changes with the arrival of the signal. The term "measure" in the previous sentence must now be interpreted to mean that the readout system must be influenced by such an operator Z which is constant in the absence of a signal. In the Schroedinger representation, this implies that

$$dZ/dt = \partial Z/\partial t - i[Z,H_D] = 0,$$

where H_D is the free Hamiltonian of the oscillator. That the readout system must depend on Z can now be interpreted to mean that the full Hamiltonian of the readout plus oscillator must have the form

$$H = H_D + \varepsilon Z\ R + H_R$$

where H_R is the free readout Hamiltonian and R is some function of the readout variables. But we now find that Z is still a constant in the absence of any further interactions with the gravity wave.

$$\begin{aligned} dZ/dt &= \partial Z/\partial t - i[Z,H] \\ &= \partial Z/\partial t - i[Z,H_D] - \varepsilon i[Z,Z]R \\ &= 0. \end{aligned}$$

If Z is chosen as the Hollenhorst type analysis suggests it should be, then Z turns out to be independent of the readout system as well. Any changes in Z discovered by its affect on the readout must originate from the gravity wave, and not from the readout system.

We have therefore succeeded in solving both the problem of readout back reaction, and quantum sensitivity at one stroke. All we need do is to find some operator Z which is time independent in the Heisenberg representation for the free oscillator uncoupled to either the gravity wave or the readout system. We must now couple the oscillator to the readout system by means of this operator sufficiently strongly that the readout system can unambiguously determine the value of Z (i.e. the eigenstate of Z which the oscillator is in). This process will not change that eigenstate. One can now calculate, a la Hollenhorst, the probability that a given gravity wave will cause the system to change its eigenstate. One can then continue measuring Z to see if the state has changed or not.

There are a number of obvious questions raised by the above, namely: do any such Z exist which are sufficiently simple that they can be realised for realistic systems; can sufficiently strong couplings be obtained to enable one to unambiguously determine Z; and finally, what happens if the real system deviates from the ideal scheme outlined above?

I will examine these questions one at a time. The simplest, time independent operator associated with a free harmonic oscillator is the energy of that oscillator. Figure 5 gives an example of a readout system, in this case the pivoted bar connecting the capacitor and inductor, acting as a readout system for an electric L-C circuit oscillator. By adjusting the length of the inductor and of the capacitor gap (or equivalently the distances from the pivot to the capacitor and the inductor) one can make the interaction between the L-C circuit and the bar via the energy in the L-C circuit. In particular we have the electromagnetic energy in the circuit as

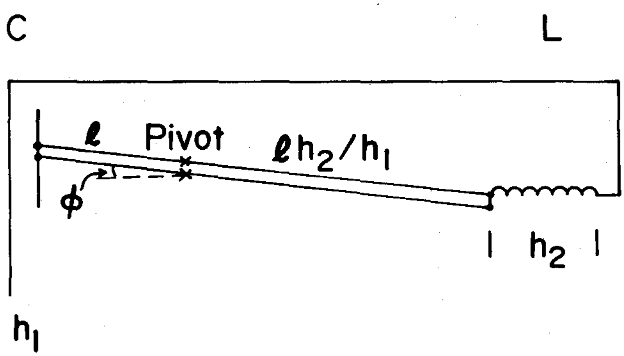

Fig. 5. Quantum non demolition readout of energy of electronic circuit coupled to mechanical bar.

$$E_{EM} = q^2/2C + \pi^2/2L,$$

where π is the total flux within the inductor, and is the momentum conjugate to the charge q on the capacitor. Now both L and C vary inversely as their length and plate separation respectively. The pivoted bar is arranged so that L/C is independent of the angle, ϕ, of the bar. We have as the total energy

$$q^2/2C(\phi) + \pi^2/2L(\phi) + J^2/2I,$$

where I is the moment of inertia of the bar, J is its angular momentum and ϕ its angular displacement. Because of the arrangement, we can write

$$C(\phi) = (1 + f(\phi))^{-1} C(o)$$

$$L(\phi) = (1 + f(\phi))^{-1} L(o)$$

for some $f(\phi)$ to give us

$$H = H_D + f(\phi) H_D + J^2/2I.$$

If, as in the diagram, $f(\phi)$ is quadratic in ϕ (at least for small ϕ) the frequency of oscillation of the bar will depend on the square root of the energy of the L-C circuit. A sufficiently accurate measurement of the bar's frequency will therefore give the free energy of the circuit (i.e. its energy at $\phi = 0$).

In principle, by making I sufficiently small, the bar's frequency can be made arbitrarily high, allowing an accurate measurement of that frequency to be made in an arbitrarily short time. Thus a measurement of the free energy of the oscillator can be made in an arbitrarily short time.

As we have heard about the Stanford gyroscope experiments here from Lipa [20], I would like to comment that the readout system proposal for that experiment is also of this kind. The coupling there is to the London magnetic moment induced in a perfect superconducting rotating sphere by means of a wire loop near the equator of the sphere. The voltage around the loop is proportional to the changes in the magnetic moment in the direction orthogonal to the loop, which correspond to changes in the angular velocity and thus in the angular momentum in that direction. Now the angular momentum is an operator of just the required type, namely, in the absence of interactions with the readout, or of other external torques, it is a constant. As would be expected, this system is most sensitive to external torques when the sphere has high total angular momentum but with the component perpendicular to the loop equal to zero.

A final simple operator associated with a harmonic oscillator which is constant in the absence of interactions was first pointed out by Thorne et al [21]. Essentially this is the initial position operator for the free oscillator,

$$X = q \cos \Omega t - (p/\Omega) \sin \Omega t.$$

Because of the free equations of motion for q and p

$$\dot{q} = p \quad \text{and} \quad \dot{p} = -\Omega^2 q$$

we have $dX/dt = 0$ as required. This quantity is therefore a suitable candidate for an optimally measurable quantity.

Is it possible to design a readout system to measure this quantity? The answer is yes. Borrowing techniques used in audio microphones we can set up a system as shown in figure 6. The movable central plate of a three plate capacitor is mechanic-

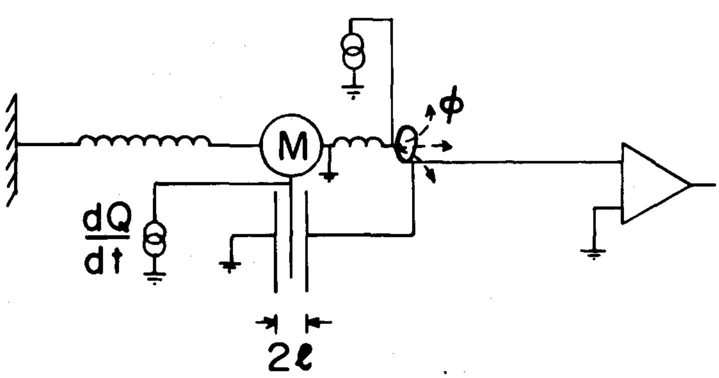

Fig. 6. Full quantum non demolition readout on mechanical oscillator.

ally connected to the oscillator mass, and a time dependent charge $Q(t)$ is placed on this plate. One of the other fixed plates is grounded and the other is connected to a loop of wire immersed in a non uniform magnetic field (like the coil of a loud-speaker). The flux, Φ, through the loop will depend on the position x of the oscillator mass, and will also be assumed to be time dependent.

The voltage across the fixed capacitor plates is given by

$$V_p = Q(t)(\ell+x)/A + 2e\ell/A \ ,$$

where 2ℓ is the separation of the plates, the movable plate is centered when the oscillator is in equilibrium, and e is the charge on the ungrounded fixed plate.

Similarly, the voltage V_c across the coil will be given by

$$V_c = (d/dt)\ \Phi(t,x) + Li \simeq \partial\Phi_0(t)/\partial t + [\partial\Phi_1(t)/\partial t]x + \Phi_1(t)dx/dt + Ldi/dt$$

where Φ_0 and Φ_1 are the first two terms in the Taylor expansion of Φ, the flux through the coil, with respect to x, L is the self inductance of the coil which is assumed to be independent of x, and $i = de/dt$. The total voltage across the capacitor and coil is therefore

$$V = (Q(t)\ell/A + 2e\ell/A + \partial\Phi_0/\partial t + Rde/dt + Ld^2e/dt^2)$$
$$+ (Q(t)/A + \partial\Phi_1/\partial t)x + \Phi_1(t)dx/dt,$$

where R is any stray resistance in the system.

The equations of motion for the oscillator are

$$M\ddot{x} = -kx + Q(t)e/A\ell - \Phi_1(t)de/dt - \Phi_1(t)i.$$

If we define the generalised momentum and coordinate of the oscillator by

$$p = \sqrt{M}\dot{x} + \Phi_1(t)e/\sqrt{M},$$

we find that the equations of motion for the oscillator are derivable from a Hamiltonian (i.e. p really is the conjugate momentum to q) and the voltage across the capacitor and inductor are

$$V = Q(t)\ell/A + 2e\ell/A + \partial\Phi_0/\partial t + Ld^2e/dt^2 + Rde/dt$$
$$+ \Phi_1{}^2(t)e/M + (Q(t)/A + \partial\Phi_1/\partial t)x + \Phi_1(t)p.$$

Now let us choose

$$\Phi_1(t) = (-f(t)/\Omega)\sin \Omega t$$

$$Q(t) = 2f(t)\cos \Omega t - (1/\Omega)(\partial f/\partial t)\sin \Omega t$$

where f(t) is some conveniently chosen function. The coupling of the readout to the oscillator is then solely in terms of the quantity $X = q \cos \Omega t - (p/\Omega) \sin \Omega t$ as required.

These kinds of couplings via variables of the type X have been extensively analysed by Thorne et. al. in a recent paper [22]. Any interested reader is referred to that paper for further analysis.

We have therefore answered the question as to whether or not such optimal readout techniques can actually be devised for realistic systems. For further comments on the theoretical aspects of these systems, called Quantum Non Demolition Readouts (QNDR), the reader is referred to a previous paper of mine [23].

We are now left with the other two questions, namely what are the effects of other external noise sources on the system (e.g. a thermal bath), and what are the effects of a less than optimal readout system?

The coupling of the detector to other external noise fields will have two effects. The first is that these fields will in general tend to damp the oscillator and thus alter its equations of motion. The mechanism for this is easily understood. The oscillator will act as a source of these fields if it is coupled to them. However, having created these fields, the oscillator will itself be affected by these fields it has created. This back reaction of the oscillator on itself alters the equations of motion of the oscillator, primarily by introducing a damping term into the equations of motion in the case of simple couplings. Secondarily, these external fields will exert forces on the oscillator, either because the states of these fields are populated, or because their vacuum fluctuations will drive the oscillator. (These vacuum fluctuation driving terms are necessary in order to maintain the commutation relations for p, q of a damped oscillator [24].

Since the oscillator's equations of motion are affected, quantities which were constants of the motion for the free oscillator are no longer constants of the motion. Furthermore, the driving terms will act as noise, and unless the gravity wave signal is much larger than the noise, the signal will be undetectable. There is no way that this noise can be eliminated except by weakening the coupling of the detector to the external fields, and by attempting to reduce the population of the states of those fields to as small a level as possible (e.g. decrease the temperature of the environment of the oscillator).

In Appendix C I have analysed a harmonic oscillator detector coupled to external damping fields which are thermally populated. It is found there that although the noise due to these fields does decrease the sensitivity and increase the probability of a false detection, the detection level can still be made much less than the "quantum limit", as long as the coupling of the detector to the amplifier is sufficiently strong, and as long as the measuring time is made much less than the damping time of the oscillator.

In the appendix, the coupling of the oscillator to the thermal bath results in a damping of the oscillator, and a shift in the resonant frequency. The quantity corresponding to $q \cos \Omega t - (p/\Omega)\sin \Omega t$ that one must couple to in the case of an undamped oscillator becomes

$$\tilde{X} = q \cos \tilde{\Omega}t - \left[(p + \sigma q/2)/\tilde{\Omega}\right] \sin \tilde{\Omega}t$$

instead where $\tilde{\Omega}$ is the shifted frequency, and σ the damping coefficient. Furthermore, this quantity is not strictly conserved by the time evolution of the system. Rather this quantity is damped as $e^{-\sigma t}$ by the back reaction of the coupling to the thermal bath. In addition, the thermal bath acts as a random force on the oscillator, which excites $\tilde{X}$ so that its squared uncertainty

$$\Delta\tilde{X}^2 = \langle\tilde{X}^2\rangle - \langle\tilde{X}\rangle^2$$

in the short term increases linearly with time. Over time periods which are long with respect to the damping time, the equilibrium between the damping and the random forces due to the thermal bath lead to

$$\Delta X^2 \sim (T/\Omega + 1/2)/2\Omega.$$

The T/Ω term is due to the thermal noise, while the 1/2 is due to the vacuum fluctuations in the thermal bath.

Over long time periods, the gravity wave must cause changes in $\tilde{X}$ at least as large as $\Delta\tilde{X}$ in order to be detectable, which, for $T = 0$, is just the usual "quantum limit". However, for times much less than the damping time, the random thermal and vacuum fluctuations do not have a sufficient time to cause large fluctuations in $\tilde{X}$, leading to an improvement in detection level by a factor of $(\tau/t_{damping})^{\frac{1}{2}}$ where τ is the measuring time, and $t_{damping}$ is the damping time of the oscillator. This result is independent of the strength of the coupling of the readout system to $\tilde{X}$ as long as that coupling strength is sufficiently large so that for

$$|\langle x\rangle| \simeq [(1/2\Omega)(T/\Omega + 1/2)\tau/t_{damping}]^{1/2}$$

more than one quantum is generated in the readout system in the measuring time. In other words, the coupling must be suffiently strong so that a minimum detectable change in $\tilde{X}$ has a measurable effect on the readout system.

Although derived in the appendix for a specific model readout system, and a simple model thermal bath, the above results are expected to hold in general for any such system.

The final question one can ask is what is the effect of a non-ideal coupling to the readout system? Let us write the Hamiltonian for our model system as

$$H = H_D + \varepsilon ZR + H_R .$$

The equation of motion for any readout variable ψ can be given as

$$d\Psi/dt = i[H_R,\Psi] + i\varepsilon Z[R,\psi].$$

Neglecting the natural time development of Ψ (i.e. assuming $[H_R,\Psi] = 0$) we have that the change in Ψ in a time δt is

$$\delta\Psi \simeq \varepsilon\langle Z\rangle\langle i[R,\Psi]\rangle\delta t.$$

By a reading of Ψ one can therefore infer a value of Z by

$$\langle Z\rangle \simeq \delta\Psi/(\varepsilon\delta t\langle i[R,\Psi]\rangle .$$

Now Ψ has an uncertainty $\Delta\Psi$ giving an uncertainty in the inferred value of Z of

$$\Delta Z \simeq \Delta\Psi/(\varepsilon\delta t\langle i[R,\Psi]\rangle .$$

But we also have the quantum uncertainty relation in the readout system

$$\Delta\Psi \; \Delta R \gtrsim \langle i[R,\Psi]\rangle \quad ,$$

from which we obtain $\Delta Z \gtrsim (\varepsilon\Delta R\delta t)^{-1}$.

Now, the change in a quantity Y associated with the harmonic oscillator due to the interaction with the readout is given by

$$\delta Y \sim i\varepsilon\langle[Z,Y]\rangle\langle R\rangle\delta t \ ,$$

and the uncertainty in Y caused by the uncertainty in <R> is

$$\Delta Y \sim \varepsilon \ \langle i[Z,Y]\rangle \ \Delta R\delta t$$

$$\gtrsim \langle i[Z,Y]\rangle/\Delta Z.$$

This gives us $\Delta Y \ \Delta Z \gtrsim \langle i[Z,Y]\rangle$.

We see that the two uncertainties in Y and in Z have two conceptually different causes. The uncertainty, ΔZ, is caused by an insufficiently strong coupling to the readout to allow one to read Z any better than ΔZ due to the quantum uncertainty in the readout. This does not mean that the oscillator is in a state with a spread in Z values, only that the readout cannot differentiate well enough. On the other hand, the ΔY is that caused by the uncertainty in the back reaction of the readout system on the oscillator. These two uncertainties - the readout and the back reaction uncertainties - are also related by the usual Heisenberg uncertainty relation. In addition, for any state of the oscillator, one has the usual exact uncertainty relations as derived in most textbooks on quantum mechanics. We therefore see how the quantum uncertainties in the readout system maintain the uncertainty relations of the measured system, as was first pointed out by Heisenberg in his microscope gedanken experiment [25].

The prescription given for an ideal Q.N.D.R. measurement is that Z is to be chosen so that dZ/dt is zero in the absence of any interaction with the readout. This implies that Z will not depend on other conjugate variables whose uncertainty is increased by the interaction with the readout. There is thus no limit on the accuracy with which Z can be measured. If, on the other hand, Z depended on some other variable Y in its time development we would have say $dZ/dt = \alpha Y$. Now in a time δt, Y would become uncertain because of its interaction with the readout system by an amount

$$\Delta Y \gtrsim \langle[Z,Y]\rangle/\Delta_R Z,$$

where $\Delta_R Z$ signifies the readout uncertainty of Z. We would have this Y uncertainty produce an uncertainty in Z of order

$$\Delta_Y Z \sim \alpha \ \Delta Y \ \delta t \gtrsim \alpha \ \langle[Z,Y]\rangle \ \delta t/ \ \Delta_R Z.$$

The total uncertainty in Z, which is a combination of the readout uncertainty and that due to the uncertainty of Y, cannot be made arbitrarily small. If one couples the readout system to the oscillator more strongly to decrease $\Delta_R Z$, one increases the uncertainty due to the back reaction. The total uncertainty

$$\Delta Z \sim \Delta_R Z + \Delta_Y Z$$

$$\gtrsim \Delta_R Z + \alpha\langle i[Y,Z]\rangle \delta t/\Delta_R Z$$

has a minimum value

$$\Delta Z \gtrsim (\alpha\langle [Y,Z]\rangle\ \delta t)^{\frac{1}{2}} .$$

For example, if we choose Z to be the position variable q, then Y will be the momentum p, and i[Y,Z] is unity, giving us

$$\Delta q \gtrsim (\delta t)^{\frac{1}{2}} .$$

Over measuring time of the order of or longer than the period of the oscillator, we find,

$$\Delta q \gtrsim \Omega^{-\frac{1}{2}}$$

which leads to the usual quantum limit that the gravity wave must produce a change in amplitude of at least $\Omega^{-\frac{1}{2}}$ to be detectable.

On the other hand, even if one does not demand exact quantum non demolition readout (QNDR), where the quantity readout is a constant of the motion, one can still do much better than this quantum limit. An example of such an approximately QNDR quantity is the time average of $2q \cos \Omega t$. We have

$$\int 2q \cos\Omega t \; dt \;\; \simeq \int (q \cos\Omega t - (p/\Omega) \sin \Omega t) \; dt.$$

By coupling to the time average of $2q \cos\Omega t$ one should be able to do almost as well as by coupling to the constant quantity $q \cos\Omega t - (p/\Omega) \sin\Omega t$. This will be true, however, only if the readout system does not perturb this time averaged quantity.

Using $Z = 2q \cos\Omega t$, we have

$$d^2Z/dt^2 = -\Omega p \sin\Omega t + \varepsilon R \cos\Omega t.$$

As we are averaging Z over a number of cycles, it is only the low frequency components in the above equation which will be of importance. Therefore, we must design the coupling so that the readout system sees only the time averaged value of Z, and so that the oscillator sees no components of the readout system with frequency near Ω.

As an example, let us look at the capacitive readout described earlier as a part of an exact system and use it on its own. We must prevent any noise source from driving the oscillator near frequency Ω. At the same time the output from the readout is to be a time average of $2q\cos\Omega t$.

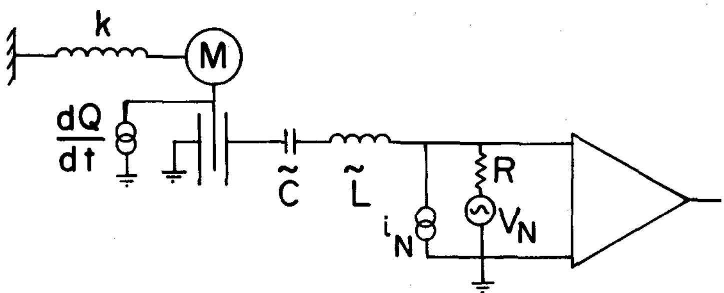

Fig. 7. Time averaged approximate quantum non demolition capacitive readout on mechanical oscillator.

The design is described in figure 7, with a charge

$$Q(t) = Q_o \cos\Omega t \cos\omega_o t$$

imposed on the center plate of the capacitor system. The voltage across the outside capacitor plates is

$$V_p = Q(t)\ell/A + 2e\ell/A + Q(t)q/\sqrt{M}A$$

The Fourier transform of V_p is (for $\omega > o$)

$$V_p(\omega) = (2\pi Q_o \ell/4A)\left[\delta(\omega-\omega_o-\Omega) + \delta(\omega-\omega_o+\Omega)\right] + e(\omega)\ell/A$$

$$+ \pi\int d\omega'\left[\delta(\omega'+\omega_o) + \delta(\omega'-\omega_o)\int dt\ q(t)\cos\Omega t\ \exp i(\omega-\omega')t\right]$$

For ω near ω_o, the last term is proportional to the time average of $q\cos\Omega t$. It is therefore only the components of V_p near ω_o which are of interest. The filter must therefore be designed to let through only these components.

From the equations of motion for the oscillator, one finds that the time averaged value of q $\cos\Omega t$ is essentially independent of the components of $e(\omega)$ near ω_o. The filter must therefore be designed so as to ensure that $e(\omega)$ has components only near the frequency ω_o in order to minimise the effect of the readout noise on the quantity being measured. Following Thorne et al., the possible noise sources, whether due to quantum zero point fluctuations, or to thermal effects, are assumed to originate as voltage and current sources at the input to the amplifier representing the rest of the readout chain. The charge components $e(\omega)$ are given by

$$e(\omega) = [2Q(\omega)/C + (Qq)(\omega)/C + i_N R + V_N]$$

$$\times\ [i\omega(R + 1/i\omega C + 1/i\omega\tilde{C} + i\omega\tilde{L})]^{-1}$$

where C is the capacticance of the two fixed readout capacitors, and (Qq) is the convolution of Q and q.

Choosing $\tilde{C}$ sufficiently small ($\ll C$) and choosing $\tilde{L} = \omega_o^2/\tilde{C}$, the charge on the capacitor due to the noise terms, i_N and V_N, will be concentrated at $\omega = \omega_o$ as required. Furthermore, the output voltage at the amplifier due to the motion of the oscillator is given by

$$V_{sig} = [(Qq)(\omega)/C][R + 1/i\omega C + 1/i\omega\tilde{C} + i\omega\tilde{L}]^{-1}.$$

Because of the peak in the denominator at $\omega = \omega_o$, the signal voltage will be proportional to the time average of $q(t)\cos\Omega t$ as noted above. This system will therefore be an acceptable approximation to an optimal readout system, with a minimum detectable signal much below the quantum limit. For further analysis of systems of this type, see the papers by Thorne et. al.

Thus we see that although ideal measurement techniques are not that difficult to achieve, even approximate techniques can do better than the "quantum limit".

III CONCLUSION

The quantum mechanics of gravity wave detectors places restrictions on the methods one can use to detect gravity waves. The most naive techniques lead to limits on the sensitivity of the detector due to the effect of the readout system on the oscillator. However, by choosing the coupling between the readout system and the oscillator appropriately, the effect of this feedback on the measurement can be eliminated.

This paper has not discussed the more stringent requirement on a system designed to not only detect the gravity wave but also to predictably infer the form of the wave which has caused a given change in a detector (something I have elsewhere called Q.N.D.R.). It has not addressed the theoretical problems inherent in designing a time dependent coupling. It does, I hope, provide an introduction to a way of thinking necessary to design and analyse detection systems in the regime where the quantum properties of the detection system become important.

APPENDIX A

I would like to present here an alternative method for looking at the interaction of a gravity wave with a solid body from that given for example by Misner, Thorne and Wheeler [26]. Although the analysis I will present is not new, [27], it does not seem to me to be widely known.

In the traditional analysis, the effect of the gravity wave on the detector is looked at as an effect of a tide producing force. The Riemann tensor of the gravity wave acts to produce a force on each particle within the detector which sets it into motion. In particular this force is equal to

$$F_i = - R_{oioj}\, x^j .$$

For a detector made of isotropic material with Lame coefficients μ,λ the equations of motion for a displacement $\tilde{u}_i$ from equilibrium are given by

$$\rho \ddot{\tilde{u}}_i = \mu \nabla^2 \tilde{u}_i + (\mu+\lambda) \; (\tilde{u}^j{}_{,j})_{,i} - \rho R_{oioj}\, x^i \quad .$$

(I have used the latin indices to designate the spatial components of any tensor. The summation convention is then over index values 1-3 and raising and lowering is done via the Euclidian spatial metric).

There exists another method for the analysis of the interaction. It essentially involves working in another coordinate system, the geodesic coordinates rather than the isometric coordinates of the above analysis. In particular I choose the coordinates in which the gravitational wave has its usual transverse traceless form [28]. Defining the gravity wave perturbation by

$$h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu} \; ,$$

we have

$$h_{o\mu} = 0,$$

$$\ddot{h}_{ij} - \nabla^2 h_{ij} = 0 \; ,$$

$$h^i_i = h_i{}^j{}_{,j} = 0.$$

In this coordinate system, a free particle initially at rest remains at rest even during the passage of the gravity wave [29] . It is thus an inertial coordinate system in that a particle will move only if acted on by external (non-gravitational) forces.

The effect of the gravity wave is to change the distances between particles. Now if the particles are part of a solid body, the interparticle spacing is determined by quantum effects (essentially by the competition between the Fermi exclusion principle and electromagnetic attractions). If the distances between particles changes, the equilibrium is upset and the particles begin to apply forces on each other, causing the body to begin to move. It is therefore the response of the body itself to the changes in metric caused by the gravity wave rather than any forces of the gravity wave on the matter which excites the detector.

Let us define $u^i(x)$ to be the displacement of the particle at x from the point x. Using standard elasticity theory, we define the strain within the body as the difference in distance between adjacent particles from their equilibrium distance. This change will be due to two causes, the presence of the gravity wave, and the relative displacements of the particles. The strain tensor ε_{ij} becomes

$$\varepsilon_{ij} = (u_{i,j} + u_{j,i} + h_{ij})/2$$

By the usual Hook's law assumption, the stress and strain are linearly related. For an isotropic material we have [30]

$$\sigma_{ij} = 2\mu\,\varepsilon_{ij} + \lambda\,\varepsilon^k_k\,\delta_{ij}\ .$$

There are now two stresses, which I will call σ_D and σ_G, due to the displacements and due to the gravity wave.

$$\sigma_{ij} = \sigma_{D\,ij} + \sigma_{G\,ij}$$

$$\sigma_{Gij} = \mu h_{ij} + \lambda\, h^k_k\,\delta_{ij}$$

$$= \mu h_{ij}\ .$$

The equations of motion for the material in the body is

$$\rho\,\ddot{u}_i = \sigma_i{}^j{}_{,j} = \sigma_{Di}{}^j{}_{,j} + \mu\, h_i{}^j{}_{,j} = \sigma_{Di}{}^j{}_{,j}\ .$$

We therefore note that within the body the equations of motion are identical with what they would have been in the absence of the gravity wave. The only effect of the wave then comes in the boundary conditions at the surface. We have

$$\sigma_{ij} \; n^j = 0,$$

where n^i is the unit normal to the surface. This gives us

$$\sigma_{D\,ij} \, n^j = -\mu \, h_{ij} n^j \; .$$

The gravity wave therefore acts like a surface traction on the body. Note that it acts only via μ , the shear modulus of the material. A material unable to sustain shears is therefore a poor candidate for a gravity wave detector.

To link this approach with the tide-producing-force approach, define

$$\tilde{u}_i = u_i - h_{ij} \, x^j$$

with the origin of the coordinates somewhere within the body. Furthermore, assume that all spatial derivatives of h_{ij} are negligible (i.e. that the wavelength of the wave is much larger than the dimensions of the body). We find

$$\rho \, \ddot{u}_i = \mu \, \nabla^2 \, \tilde{u}_i + (\mu+\lambda) \, \tilde{u}^j{}_{,j,i} + \rho \, \ddot{h}_{ij} \, x^j$$

which is equivalent to eq. A.1. Furthermore, the boundary conditions on

$$\tilde{\sigma}_{D\,ij} = \mu \, (\tilde{u}_{i,j} + \tilde{u}_{j,i}) + \lambda \, \tilde{u}^k{}_{,k} \delta_{ij}$$

are given by

$$\tilde{\sigma}_{Di}{}^j \, n_j = 0$$

The relation between the two viewpoints is thus simply a change of coordinates. It is, however, instructive to note that the usual expression applies only to the long wavelength limit. Furthermore, the explicit dependence of the motion of the body on only the shear modulus of the material is far from obvious in the usual analysis of the interaction.

APPENDIX B

In this appendix I will present a simple solvable model of an oscillator type gravity wave detector.

To make it somewhat realistic, the oscillator will be damped and will be in a thermal bath of temperature T. The readout system will be via a coupling to an external field, and will be a time independent linear coupling so that the oscillator will serve only to convert gravity wave quanta into readout quanta. This simple model will be used to demonstrate explicitly the relation between the Q of the oscillator and the signal to noise ratio, and the effect of coupling to the readout system on the signal to noise ratio.

The model will assume that the "gravity wave" is a massless, two dimensional scalar field $\Phi_{(o)}$. The thermal bath and the fields, $\Phi_{(i)}$, i>0, which damp the oscillator will also be two dimensional scalar fields, as will be the readout field, Ψ.

The Langrangian action of this model system is given by

$$\int\left\{(1/2)\sum_i [\dot{\Phi}^2_{(i)} - (\partial\Phi_{(i)}/\partial x)^2] + [\sum_i \alpha_i \dot{\Phi}_{(i)} q + \dot{q}^2/2 - \Omega^2 q^2/2 + \beta q \dot{\Psi}]\ \delta(x) + [\dot{\Psi}^2 - (\partial\Psi/\partial x)^2]/2\right\} dx\, dt$$

The equations of motion for this system can be solved exactly. In particular, the readout field Ψ depends on the *in* fields $\Phi_{(i)I}$ and Ψ_I (which obey the two dimensional massless wave equation exactly). We have

$$\Psi(t,x) = \Psi_I(t,x) - (\beta/2)(q(t-x)\theta(x) + q(t+x)\ \theta(-x)),$$

where q(t) is best expressed in terms of its Fourier transform

$$q(\omega) = \int e^{i\omega t}\, g(t)\, dt.$$

The equation of motion for q(t) is

$$\ddot{q} + (\sum_i \alpha(i)^2 + \beta^2)\ \dot{q}/2 + \Omega^2 q = \sum_i \alpha_i\ \dot{\Phi}_{(i)I}(t,o) + \beta\ \dot{\Psi}_I\ (t,o)$$

which is the equation of a damped harmonic oscillator, with the *in* Φ and Ψ fields acting as forces on the oscillator. The solution is

$$q(\omega) = + \frac{+i\omega\left(\sum_i \alpha_i\ \Phi_{(i)I}(\omega,o) + \beta\ \Psi_I(\omega,o)\right)}{\omega^2 + i\omega\ (\sum_i \alpha_i^2 + \beta^2)/2 - \Omega^2}$$

Let us now assume that we are observing the Ψ field at the point x=0. The value of the Fourier transorm of Ψ at this point is then given by

$$\Psi(\omega) = \Psi_I(\omega,o) - \frac{i\beta\omega}{2}\left(\frac{\sum_i \alpha_i \Phi_{(i)I}(\omega,o) + \beta\omega_I(\omega,o)}{\omega^2 + i(\sum_i \alpha_i^2 + \beta^2)\,\omega/2 - \Omega^2}\right)$$

As $\Psi(\omega)$ depends only on $\Phi_I(\omega,o)$ and $\Psi_I(\omega,o)$, there is no mixing of positive or negative frequencies from the *in* to the *out* states. This system therefore acts as a pure transducer with no amplification.

Note that the damping of the oscillator itself arises out of the back reaction of the emission of quanta into the various $\Phi_{(i)}$ fields and the readout field. The net Q of the oscillator therefore depends on both the strength of the coupling to the thermal bath, the "gravity wave" and to the readout field. For optimal detection one would expect that β should be sufficiently large that the oscillator decayed predominantly via emission into the readout channel, rather than into the thermal bath.

In order to proceed with this analysis, we need to design a detection strategy. Let us assume that one measures the number of particles in some mode of the Ψ field at x=0 which averages the output over some time period τ. The number operator will be of the form

$$N = C^\dagger C$$

with

$$C = \int_{\omega>o} (\omega/2\pi)^{\frac{1}{2}}\, c(\omega)\, \Psi(\omega)\, d\omega,$$

and

$$\int_{\omega>o} |c(\omega)|^2\, d\omega = 1$$

where $|c(\omega)|$ is a smooth function of width $1/\tau$ centered at $\omega=\Omega$. The normalisation factor occurring in the definition of C is appropriate for a two dimensional scalar field.

We now wish to calculate the expectation value of N under the assumption that the $\Phi_{(i)I}$ states with i>o are thermally populated with temperature T which gives

$$< \Phi^\dagger_{(i)I}(\omega,o)\ \Phi_{(i)I}(\omega',o) >_{\omega>o} = 2\pi(2T/\omega^2)\ \delta_{ij}\ \delta(\omega-\omega').$$

(The extra factor of 2 arises because there are Φ_i modes travelling in both directions). The gravity wave, $\Phi_{(o)I}$, is assumed to consist of a pulse with a broad frequency spectrum including the frequency Ω of the oscillator while the readout field is initially in the vacuum state. I will further assume that the measuring time τ is less than the decay time of the oscillator.

We obtain

$$< N > = \frac{|\beta|^2|\alpha_o|^2}{4} <D_o^\dagger D_o> + \frac{|\beta|^2}{4} \sum_{i>o} \int_{\omega>o} \omega^2 \frac{|\alpha_i|^2 \ |c(\omega)|^2 \ (2T/\omega)\ d\omega}{(\omega^2-\Omega^2)^2 + (\sum_i \alpha_i^2+\beta^2)^2 \ \omega^2/4}$$

where we define

$$D_o = i \int \sqrt{\frac{\omega}{2\pi}} \frac{c(\omega) \ \Phi_{(o)I} \ (\omega,o) \ \omega \ d\omega}{\omega^2+i\omega \ (\sum_i \alpha_i^2+\beta^2)/2-\Omega^2} \quad .$$

The integrals will be dominated by the pole near $\omega=\Omega$. We therefore obtain

$$<N> \approx \frac{\Omega}{4} |\beta|^2|\alpha_o|^2 \ |c(\Omega)|^2 \quad <\Phi^\dagger_{(o)I}(\Omega,0) \ \Phi_{(o)I}(\Omega,0)> + \frac{\pi|\beta|^2(\sum_{io} \alpha_i^{\,2}) \ |c(\Omega)|^2 (2T/\Omega)}{4 \ (\sum_i \alpha_i^2 + \beta^2)} \quad .$$

The first expectation value is just the number quanta in the incoming wave at frequency Ω , i.e.

$$< \Phi^\dagger_{(o)I} \ (\Omega,0) \quad \Phi_{(o)I}(\Omega,0) \ > \approx \frac{2\pi}{\Omega} n(\Omega)$$

to finally give

$$< N > = (1/2i)\pi\beta^2|c(\Omega)|^2 \ [|\alpha_o| \ n(\Omega) \ + \sum_{i>o}(\alpha_i^2T/\Omega)/(\sum_i \alpha_i^2 \ + \ \beta^2)]$$

The signal to noise ratio is now given by

$$S/N = \alpha_o^{\,2} n(\Omega)(\sum_i \alpha^{\ 2} + \beta^2)/\sum_{i>o}\alpha_i^{\,2}(T/\Omega) .$$

We see that the larger $|\beta^2|$ is (i.e. the stronger the coupling to the readout), the better is the signal to noise ratio. The essential reason is that the thermal fluctuations do not have a chance to build up the amplitude of oscillation before they decay. On the other hand, the gravity wave impulse will excite the oscillator by the same amount, no matter what the decay time, as long as the pulse is shorter than the decay time or measuring time.

Thermal noise is not, however, the only uncertainty. If the number <N> is of order unity or less, the Poisson fluctuations in the count will introduce uncertainty. This limit is given by

$$(\pi\beta^2/2)\,|\,c(\Omega)\,|^2\ \alpha_o^2\ n(\Omega)\ >\ 1.$$

Because of the definition of $c(\Omega)$ we have

$$|c(\Omega)|^2 \sim \tau\ ,\ \text{or}\quad \beta^2\tau\alpha_o^2\ n(\Omega)\ \gtrsim 1.$$

This is maximised by letting $\beta^2\tau \sim 1$. (If $\beta^2\tau > 1$, the above derivation of <N> fails and no advantage is gained). The limit $\alpha_o^2\ n(\Omega)\ \gtrsim 1$ is essentially the so-called quantum limit.

If, for this type of transducer, we optimise $|\beta|^2$ and τ for maximum sensitivity (i.e. $|\beta|^2\tau{\sim}1$), then the thermal noise depends on $T\sum_{i>o}\alpha_i^2$. Assuming $|\alpha_o|^2 << |\alpha_i|$ (as is certainly true for any known detector), this product is just proportional to T/Q where Q is the quality factor of the oscillator in the absence of any readout coupling. Furthermore, if β is sufficiently large ($\beta^2 > \sum_i \alpha_i^2$), the thermal noise goes as $1/\beta^2$. The optimum strategy therefore becomes to make the measuring time approximately equal to the decay time of the oscillator, to make T/Q as small as possible both by decreasing the temperature and by decreasing the coupling of the oscillator to any spurious fields, and to make the coupling of the oscillator to the readout system as strong as possible.

The above are of course well known, but it is reassuring to see these conclusions follow from a simple, exactly solvable model.

APPENDIX C

This appendix presents a detailed analysis of a model quantum non demolition readout system coupled to a damped harmonic oscillator, which is under the influence of a signal field and of thermal noise sources. The system will be mathematically idealised so as to make it exactly solvable, but will retain enough features of a realisable system to act as a guide to the behaviour of such a system.

The oscillator is assumed to have momentum and position coordinates p and q which are coupled to a set of one-spatial dimensional scalar fields Φ_i. These fields will provide the damping of the oscillator and the source of the thermal noise. Also, one of the fields, Φ_o, will be the signal channel. (i.e. it is signals present in this channel which we will want to detect). The "measurements" on the oscillator will be made by means of a "readout field" Ψ. For simplicity I will assume that the amplitude of Ψ is directly measurable by some means which I will not analyse further. All of the Φ_i fields, the oscillator and the Ψ field will be considered to be fully quantum mechanical.

The measurements on the readout field Ψ will be taken to be measurements (in the quantum sense) of the amplitude operators

$$A_{f,T} = (1/2) \int f(t-T) \cos \omega_o t \, \Psi(t,x_o) \, dt$$

at some point $x_o > 0$. (For mathematical convenience we can take $x_o = 0^+$) Define the positive frequency function h(t) by its Fourier transform

$$h(\omega) = (1/2i\omega) \, (f(\omega-\omega_o) + f(\omega+\omega_o)) \, \theta(\omega)$$

I will assume that h(t) is normalised so that

$$-i \int \dot{h}^*(t) \, h(t) \, dt = 1$$

where the dot indicates the time derivative. The operator $A_{f,T}$ is then equal to

$$(a^\dagger_{h,T} + a_{h,T}) \, ,$$

where $a_{h,T}$ is the annihilation operator associated with the mode h(t-T) (i.e. the mode h centered at time T). f(t) will be assumed to be a smooth real function of width τ centered at time t=0, while its Fourier transorm $f(\omega)$ will have width of order $1/\tau$ centered at $\omega=0$. τ will in this case represent the averaging time of the measurement which will be assumed to be much longer than the oscillator period, but less than the decay time of the oscillator.

$A_{f,T}$ is thus a measure of the amplitude of the $\cos\omega_o t$ component of Ψ at time T averaged over time τ. It is Hermitean and thus a measurable quantity in the quantum sense. I will leave the measurement technique unspecified (one has to stop somewhere).

Because of the coupling of the Φ_i fields of the oscillator, the presence of thermal noise, or of a signal in the Φ_i fields, will change the oscillator coordinates. Furthermore, because of the coupling of the Ψ field to the oscillator, changes will thus be produced in Ψ . The change in the value of $A_{f,T}$ with T will then give a measure of the signal (or noise) in the Φ_i fields.

Having described the system to be used, we can now set up the model to show that one can in principle set up a quantum non demolition readout (i.e. one in which the readout can contribute negligible noise to the measuring process). The full action for this system will be given by

$$\dot{p} = -\Omega^2 q - \sum_i \alpha_i q/2 - (\beta^2/2)\cos\omega_o t[(\cos\tilde{\Omega}t - \sigma\sin\tilde{\Omega}t/\tilde{\Omega})\ (d/dt)(\cos\omega_o tX)]$$

$$- \sum_i \alpha_i {}^{o}\dot{\Phi}_i - \beta\cos\omega_o t\ [\cos\tilde{\Omega}t - \sigma\sin\tilde{\Omega}t/\tilde{\Omega}]\ {}^{o}\dot{\Psi}$$

$$\dot{q} = p - (\beta^2/2)\cos\omega_o t\ [\sin\tilde{\Omega}t/\tilde{\Omega}\ d(\cos\omega_o tX)/dt] - \beta\cos\omega_o t\ [\sin\tilde{\Omega}t/\tilde{\Omega}\ {}^{o}\dot{\Psi}]$$

Note that if $\beta = 0$ (no readout), the equations for p and q are those of a damped harmonic oscillator with damping term $\sum_i \alpha_i^2/2 = 2\sigma$ and forcing term $-\Sigma\ \alpha_i\ {}^{o}\dot{\Phi}_i$.

Instead of solving for p, q. it is much simpler to solve for X. We find

$$\dot{X} = -\sigma X + (\sin\tilde{\Omega}t/\tilde{\Omega})\sum_i \alpha_i\ {}^{o}\dot{\Phi}_i .$$

Notice that Ψ, the readout field depends only on the variable X while X depends only on the *in* fields ${}^{o}\Phi_i$. This demonstrates the exact quantum non-demolition nature of this interaction.

It will now be simpler to examine the Fourier transform of the field Ψ at $x = 0+\varepsilon$. Defining

$$\Psi(\omega) = \int e^{i\omega t}\Psi(t,0+\varepsilon)\ dt$$

we have

$$\Psi(\omega) = {}^{o}\Psi(\omega) + (\beta/4)(X(\omega+\omega_o) + X(\omega-\omega_o)) .$$

To simplify future discussion, I will assume that if $\omega>0$, the term proportional to $X(\omega+\omega_o)$ can be neglected, while for $\omega<0$, $X(\omega-\omega_o)$ may be neglected. (This essentially assumes that the oscillator decouples from the Φ_i fields at sufficiently high frequencies). Solving for X we finally have for $\omega>0$

$$\Psi(\omega) \simeq {}^{o}\Phi(\omega) + (\beta/8)\Sigma\ (\alpha_i/\tilde{\Omega})(i/(2(\omega-\omega_o) + \sigma))$$

$$\times \left[(\omega-\omega_o+\tilde{\Omega})\ {}^{o}\Phi_i(\omega-\omega_o+\tilde{\Omega}) - (\omega-\omega_o-\tilde{\Omega})\ {}^{o}\Phi_i(\omega-\omega_o-\tilde{\Omega})\right]$$

and we find

$$A_{f,T} = \int d\omega\ f^*(\omega) e^{i\omega T}[\Psi(\omega+\omega_o) + \Psi(\omega-\omega_o)]/4\pi$$

$$+ (\beta/8\tilde{\Omega})\sum_i \alpha_i \int d\omega\ f^*(\omega) e^{i\omega T}\ [(\omega+\tilde{\Omega})\ {}^o\Phi_i\ (\omega+\tilde{\Omega}) - (\omega-\tilde{\Omega})\ {}^o\Phi_i(\omega-\tilde{\Omega})]/2\pi(i\omega+\sigma)$$

The expectation value of $A_{f,T}$ will obviously depend on the initial state of the Ψ field which we will assume to be the vacuum state (i.e. no initial Ψ particles, at least not with frequency near ω_o), and on the initial value of the ${}^o\Phi_i$ fields. In particular, by making β large enough, the effect of the ${}^o\Phi_i$ fields on the expectation value of $A_{f,T}$ can be made as large as desired. This system therefore definitely acts as an amplifier-transducer.

The important point is to calculate the noise introduced into the measurement of $A_{f,T}$ both by quantum and by thermal effects. The simplest method is to calculate the expectation value of $(A_{f,T})^2$ in the state in which there are no coherent incoming ${}^o\Phi_i$ waves, but the ${}^o\Phi_i$ states are thermally excited. In a thermal state we have the expectation value

$$< {}^o\Phi_i(\omega)\ {}^o\Phi_j(\omega') > = [(2\pi\delta_{ij}\ \delta(\omega+\omega'))/|\omega|]\quad [T/|\omega| + \theta(-\omega)]\ .$$

This equation results because for a 1 dimensional wave $\Phi(\omega)/(2\pi\omega)^{\frac{1}{2}}$ is the annihilation operator for the mode of frequency ω. The first term in the above expression is the thermal factor where T is the temperature (in units where k = h = 1) while the second is due to the quantum nature of the fields. Also, because of the real (Hermitean) nature of the fields we have

$${}^o\Phi_i^{\dagger}(\omega) = {}^o\Phi_i(-\omega)\ .$$

We also have

$$< {}^o\Psi(\omega)\quad {}^o\Psi(\omega')> = [(2\pi\delta(\omega+\omega')/|\omega|]\ \theta(-\omega)$$

as, by assumption, Ψ is initially in its vacuum state.

There are now two alternatives. One can measure the amplitude $A_{f,T}$ at one time to determine whether or not the measured value differs appreciably from that expected from the noise terms alone. The criterion here is that the expected signal must be greater than the amplitude expected due to noise alone; i.e. it must be greater than $< A_{f,T}^2 >^{\frac{1}{2}}$ where the expectation is that in the state with no signal input. We have

$$< A^2{}_{f,T} > = \int \frac{|f(\omega)|^2 d\omega}{8\pi\ |\omega+\omega_o|} + \frac{\beta^2}{64\tilde{\Omega}^2}\ \sum_i \alpha_i{}^2 \int \frac{|f(\omega)|^2(\omega^2-\tilde{\Omega}^2)}{2\pi(\omega^2+\sigma^2)\ |\omega|}\cdot\ (1+2T/|\omega+\tilde{\Omega}|)\ d\omega$$

where I have assumed that the width of $f(\omega)$ is much less than Ω. By the normalisation of $f(\omega)$, the first term is unity. The second term is dominated by the pole at ω=o, giving

$$< A^2_{f,T} > \simeq 1 + (\beta^2/2^7\tilde{\Omega}\sigma)\Sigma\alpha_i^2|f(o)|^2 \ (1 + 2T/\tilde{\Omega})$$

$$\simeq 1 + (\beta^2/2^5\tilde{\Omega}) \ |f(o)|^2 \ (1 + 2T/\tilde{\Omega}).$$

On the other hand, one expects the noise at two measurements separated by less than the damping to be correlated. This is born out by calculating the expectation value of the product of the amplitudes at two times T, T' (chosen so that $|T'-T|$ is greater than the averaging time). We find

$$< A_{f,T} \ A_{f,T'}> \simeq \ (\beta^2/2^5\tilde{\Omega})\,|f(o)|^2 \ (1+2T/\tilde{\Omega}) \ \exp - \sigma|T-T'|.$$

Because of this correlation, it is better to measure the change in the amplitude $A_{f,T}$ over a time period shorter than the decay time of the oscillator as we have

$$< (A_{f,T} - A_{f,T'})^2 > = 2 + (1 - \exp -\sigma|T-T'|)(\beta^2|f(o)|^2/2^4\tilde{\Omega})(1 + 2T/\tilde{\Omega})$$

$$\simeq 2 + (\sigma|T-T'|)(\beta^2|f(o)|^2/2^4\tilde{\Omega})(1 + 2T/\tilde{\Omega}) \ .$$

The change in A_f caused by the signal is given by

$$|< A_{f,T} - A_{f,T'}>| \ \simeq \ |(\beta\alpha_o \ f(o)/8)< \ {}^o\Phi_o(\tilde{\Omega}) + {}^o\Phi_o(-\tilde{\Omega})>|$$

To be detectable, this must be greater than the noise, from which we obtain

$$\tilde{\Omega}\alpha_o{}^2|< {}^o\Phi_o(\Omega) + {}^o\Phi_o(-\Omega)>|^2 > (2^7\tilde{\Omega}/|\beta f(o)|^2) + 4\sigma(T-T')2T/\Omega$$

From the normalisation of $h(t)$ we have

$$|f(o)|^2 \simeq \omega_o\tau$$

where τ is the averaging time. We finally have

$$(\alpha_o{}^2/4)|< \ {}^o\Phi_o(\tilde{\Omega}) + {}^o\Phi_o(-\tilde{\Omega})>| \geq \ (2^5/|\omega_o\beta\tau|^2) + (\sigma|T-T'|/\tilde{\Omega})(1 + 2T/\tilde{\Omega})$$

The ℓ.h.s. of this expression is just the change in X caused by the signal. The usual "quantum limit" would replace the r.h.s. by $(2\ \tilde{\Omega})^{-\frac{1}{2}}$. By choosing a sufficiently large β, the first term can be made negligible, while the second term can only be decreased by reducing the temperature or reducing the damping constant of the oscillator.

The measuring time T-T' must remain longer than the averaging time τ or the above analysis fails. To get an estimate of how far the present state of the art could exceed the "quantum limit", we can choose a frequency of order 1 k hz, a Q of 10^{10} and a temperature T of .1 K. Choosing T-T' (the measuring time) of one second, we find that one can just reach the quantum limit sensitivity. Thus some significant advances over present technology will need to be achieved to make such a scheme feasible.

ACKNOWLEDGEMENT

I would like to thank the National Research Council of Canada and the Alfred P. Sloan Foundation for support of this work and the organisers of this school for the travel grant which enabled me to attend.

NOTES AND REFERENCES

1 The first person to seriously worry about this quantum limit and think about techniques for avoiding this limit was V. Braginsky. See Braginsky, V.B., and Manukin, A.B., in *Measurement of Weak Forces in Physics Experiments*, University of Chicago Press, Chicago, 1977 ed. Douglas, D.H. See also
Braginsky V.B., and Vorontsov, Y.I., *Usp.Fiz.Nauk.* 114, 41, (1974) [*Sov.Phys. Usp.* 17, 644 (1975)]:
Braginsky, V.B., Vorontsov, Y.I., Krivchewkov, V.D., *Zh.Exsp.Tesp.Teor.Fiz.* 68,55, (1975) [*J.E.T.P.* 41,28,(1975)].

2 One would expect the strongest sources to be highly nonlinear in the source region, but to give a deviation from flatness as seen at infinity of less than unity by the time one arrived at the radiation zone, i.e. ~ one wavelength from the source. Actual sources are probably much weaker than this.

3 As the collapse time for a solar mass black hole is about $10^{-5}$to 10^{-4}sec., the spectrum of gravity waves should extend to 10 Khz, with a reasonable strength of wave emitted for an asymmetric, rapidly rotating final collapse stage.

4 See for exaple Tamman, G.A., "Statistic of Supernovae in External Galaxies" in *Eighth Texas Symposium on Relativistic Astrophysics* ed. Papagiannis M.D., New York Acad. Sc.,(N.Y.)1977 who derives a figure of about 1 per 10 years for our galaxy.

5 This uses the classical estimate of the energy in a gravity wave given, for example, in Misner, C., Thorne, K., and Wheeler J., *Gravitation* Freedman (N.Y.) 1975 p.955f.

6 Heffner, H., *Proc. I.R.E.* 50, 1604 (1962)

7 Haus, H.A., and Mullen, J.A., *Phys.Rev.* 128, 2407.

8 The (1962) "well known" commutation relation between number and phase is not exact and not derivable from quantum theory because of the non existence of an operator corresponding to phase conjugate to N. See for example Carruthers P., Nieto, M.M., *Rev.Mod.Phys.* 40, 411 (1968) for a discussion of some of these problems.

9 These field normal modes are the C-number solutions of the wave equations for Ψ and Φ under the assumption of no coupling between the fields. See for example Bjorken, J., Drell, S., *Relativistic Quantum Fields* McGraw Hill (N.Y.) 1964.

10 See reference 9.

11 Even in the case of non linear interactions, the commutation relations place strong restrictions on the form of the S-matrix which maps the ingoing states to the outgoing states.

12 Von Neuman J., in *Mathematical Foundations of Quantum Mechanics* (Tr. Beyer,R.T.) Princeton University Press (1955) discusses the problem of breaking the chain of analysis in any quantum measurement process.

13 Einstein A., *Phys.Zeits.* 18, 121 (1917)

14 Paper in preparation.

15 Hollenhorst, J.N., *Phys.Rev.D.* 19, 1669 (1979)

16 Helstrom C.W., *Quantum Detection and Estimation Theory* Acad. Press (N.Y.) 1976

17 This is of course the property which sets quantum mechanics off from classical mechanics, that different states can have some probability of being indistinguishable.

18 The coherent states were introduced by Schroedinger E., *Z.Physik.*,14, 664 (1926), and are minimum uncertainty ($\Delta p \Delta q = h/2$) states. They are essentially eigenstates of the annihilation operator. See also Glauber, R.J., *Phys.Rev.* 131, 2766 (1963)

19 This analysis was actually derived by Thorne K., et.al. in ref 21 and Unruh W. in ref 23 before Hollenhorst's works.

20 See J. Lipa lectures in this volume.

21 Thorne K., Drever, R.W.P., Caves C.M., Zimmerman, M., and Sandberg V.D., *Phys. Rev.Lett.* 40, 667 (1978)

22 Caves, C.M., Thorne, K.S., Drever R.W.P., Sandberg V.D., and Zimmerman, M., *"On the Measurement of a Weak Classical Force Coupled to a Quantum Mechanical Oscillator I. Issues of Principle"* Cal. Tech. preprint Apr. 1979.

23 Unruh W., *Phys.Rev.D.* 19, 2888 (1979)

24 See for example the discussion in pp. 331f in Louisell W.H., *Quantum Statistical Properties of Radiation* Wiley, (N.Y.) 1973.

25 See for example the discussion in Messiah A., *Quantum Mechanics* Wiley, (N.Y.) 1966 on pp. 139-149. The argument presented in this paper demonstrates how the quantum uncertainties in the readout system preserve the uncertainties of any variables being measured.

26 See Misner, Thorne, Wheeler (ref 5) on p. 1031f.

27 Carter B., Quintana H., *Phys.Rev.D.* 16, 2928 (1977) Dyson F., *Ap. J.* 156, 529 (1969).

28 See Misner, Thorne, Wheeler (ref 5) on p. 946f.

29 The geodesic equations for the spatial components of the position

$$d^2X^i/d\lambda^2 + \Gamma^i_{\mu\nu}(dX^\mu/d\lambda)(dX^\nu/d\lambda) = 0$$

will maintain X^i constant if $dX^i/d\lambda$ is initially zero for all i since Γ^i_{oo} depends only on h_{ot}.

30 μ and λ are the usual Lame coefficients for an isotropic medium.