Physics 501-21
Assignment 3

1.) To show the unusual power of "post-selected" quantum mechanics, Aharonov developed the "Mean King" puzzle. A Physicist is shipwrecked on an island which is ruled by a King. He loves cats and thus hates physicists. But to be fair, he gives the physicists a problem. He has on his island a laboratory in which has all of the greatest and latest equipment. He gives this problem.

He hands the physicists a spin 1/2 particle which is in an isolated chamber such that he can assure the physicists that the free Hmiltonian is 0. Now he says that the physicists can use the lab and that spin 1/2 particle. After the physicist has done whatever he wants, the King takes that spin 1/2 particle and makes a measurement of either the spin in the x direction, the spin in the y direction, or the spin in the z direction, but does not tell the physicist which he measured. He then hands the physicist back the particle and the physicists can again do whatever he wants. The physicists is then called into the throne room, and told which spin the king measured, and the physicist is asked what the value was that was measured. This is repeated many times. If the physicist ever gets the wrong answer, she is killed. How can she survive?

Note that if there were just two, say spin the x or y directions, it would be easy. Before hand hand she would prepare the state in the x direction. Afterwards she would measure it in the y direction. If the king measured it in the x direction, the initial state would tell her what his answer would have been. If the king measured it in the y direction, the final measurement would tell her what the answer was. But with three, this clearly would not work. (If she carried out the above, and the king measured it in the z direction, there would only be a 50% chance of getting the answer) and thus she would have a good chance of dying. (1/6 probability on each repetition).

The answer is in the paper

Aharonov, VAidman, Albert Phys Rev Letters 58, 1385 (1987)

Explicitly verify the various claims in the paper. Eg, prove that the states in eqn2 are correct and that the outcomes in the table above eqn 2 are right.

While if there were only two choices one could solve this by doing a pre measurement on the particle of say σ_x and a post measurement of σ_y (If the King had picked σ_x the pre measurement would tell what it is, while if the King picked σ_y the post measurement would tell what it was.) This however does not work for σ_z as well. That would give only a 50-50 chance of both possible values. However, the only possible way would be to entangle the particle with another two level particle, and do post measurements on the two particles.

The paper gives the procedure.

a) Prepare the two spins in some Bell state. The one they advocate is the state

$$\Psi = \frac{1}{\sqrt{2}}(|\uparrow\rangle\,|\uparrow\rangle + |\downarrow\rangle\,|\downarrow\rangle) \tag{1}$$

Now measure some operator A which has the eigenstates

$$|\phi_1\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_e|\uparrow\rangle + \frac{1}{\sqrt{2}}(|\uparrow\rangle_e|\downarrow\rangle e^{i\pi/4} + |\downarrow\rangle_e|\uparrow\rangle e^{-i\pi/4})\right] \tag{2}$$

$$|\phi_2\rangle = \frac{1}{\sqrt{2}}\left[|\uparrow\rangle_e|\uparrow\rangle - \frac{1}{\sqrt{2}}(|\uparrow\rangle_e|\downarrow\rangle e^{i\pi/4} + |\downarrow\rangle_e|\uparrow\rangle e^{-i\pi/4})\right] \tag{3}$$

$$|\phi_3\rangle = \frac{1}{\sqrt{2}}\left[|\downarrow\rangle_e|\downarrow\rangle + \frac{1}{\sqrt{2}}(|\uparrow\rangle_e|\downarrow\rangle e^{-i\pi/4} + |\downarrow\rangle_e|\uparrow\rangle e^{i\pi/4})\right] \tag{4}$$

$$|\phi_4\rangle = \frac{1}{\sqrt{2}}\left[|\downarrow\rangle_e|\downarrow\rangle - \frac{1}{\sqrt{2}}(|\uparrow\rangle_e|\downarrow\rangle e^{-i\pi/4} + |\downarrow\rangle_e|\uparrow\rangle e^{i\pi/4})\right] \tag{5}$$

The four unit vectors $|\uparrow\rangle_e|\uparrow\rangle$, $|\downarrow\rangle_e|\downarrow\rangle$, $\frac{1}{\sqrt{2}}(|\uparrow\rangle_e|\downarrow\rangle e^{i\pi/4} + |\downarrow\rangle_e|\uparrow\rangle e^{-i\pi/4})$, and $\frac{1}{\sqrt{2}}(|\uparrow\rangle_e|\downarrow\rangle e^{-i\pi/4} + |\downarrow\rangle_e|\uparrow\rangle e^{i\pi/4})$ are all orthogonal to each other. (the third an fourth are perphaps the non-obvious ones, but the inner produce is proportional to $e^{i\pi/2} + e^{-i\pi/2} = i + (-i) = 0$. The first and second pairs are each sum and differences of mutually orthogonal vectors and are thus orthogonal to each other.

Now one simply needs to check on the expectation of the various σ_i between the $|\Psi\rangle$ and the various ϕ_a states.

[Note: $\langle\mu|\,|\nu\rangle_e|\rho\rangle = \langle\mu|\,|\rho\rangle\,|\nu\rangle_e$]

First we look at the expectation value of the various projection operators onto the various eigenstates of the measurement

$$\begin{aligned}
\sigma_z &= +1 \rightarrow P_{z,1} = |\uparrow\rangle\langle\uparrow| && (6)\\
P_{z,1}|\Psi\rangle &= \frac{1}{\sqrt{2}}|\uparrow\rangle_e|\uparrow\rangle && (7)\\
\sigma_x &= +1 \rightarrow P_{x,1} = \frac{1}{2}((|\uparrow\rangle + |\downarrow\rangle)(\langle\uparrow| + \langle\downarrow|) && (8)\\
P_{x,1}|\Psi\rangle &= \frac{1}{2}((|\uparrow\rangle_e + |\downarrow\rangle_e)(|\uparrow\rangle + |\downarrow\rangle) && (9)\\
\sigma_y &= +1 \rightarrow P_{y,1} = \frac{1}{2}(|\uparrow\rangle - i|\downarrow\rangle)(\langle\uparrow| + i\langle\downarrow|) && (10)\\
P_{y,1}|\Psi\rangle &= \frac{1}{2}((|\uparrow\rangle_e + i|\downarrow\rangle_e)(|\uparrow\rangle - i|\downarrow\rangle) && (11)\\
& && (12)
\end{aligned}$$

where $P_{z,1}$ is the projection operator onto the $\sigma_z = 1$ eigenstate. (Here $\sigma_z = 1$ means This is the eigenvalue of σ_z with eigenvalue of +1)

$|\phi_i\rangle P_{z,1}|\Psi\rangle = \frac{1}{2}$ for i=1 or 2, and is 0 for 3 or 4. since only the first two ϕ states have a $|\uparrow\rangle|\uparrow\rangle$ term in them.

Similarly $|\phi_i\rangle P_{z,1}|\psi\rangle = 0$ for 1 and 2 and $\frac{1}{2}$ for 3 or 4. Thus if the scientist measured i of 1 or 2 afterwards, they knew that σ_z was value 1, and was value -1 for 3 or 4.

Thus for each $|\phi_i\rangle$ state, the probability of some eigenvalue of σ_z is either 1 or 0. which of the states $|\phi_i\rangle$ one measured, the value of σ_z is definite.

Similarly for σ_x.

$$\langle\phi_{1,2}|\, P_{x,1}\,|\psi\rangle = \langle\phi_{1,2}|\,\frac{1}{2}(|\uparrow\rangle_e\,|\uparrow\rangle + |\downarrow\rangle_e\,|\downarrow\rangle + |\downarrow\rangle_e\,|\uparrow\rangle + |\uparrow\rangle_e\,|\downarrow\rangle) \tag{13}$$

$$= \langle\phi_{1,2}|\,\frac{1}{2}(|\uparrow\rangle_e\,|\uparrow\rangle + |\downarrow\rangle_e\,|\uparrow\rangle + |\uparrow\rangle_e\,|\downarrow\rangle) \tag{14}$$

$$= \frac{1}{2\sqrt{2}}(1 \pm \sqrt{2}(cos(\pi/4) = \frac{1}{2\sqrt{2}}(1 \pm 1) \tag{15}$$

and similarly for 3,4. For $\sigma_x = -1$, we have 2 and 4 having inner product of 1/4 while 1 and 3 have 0. Thus 1 and 3 will have probability 1 of have having value of σ_x be 1 while 2 and 4 will have probability of 1 of having σ_x be -1.

Finally, doing the same for σ_y be 1, we have

$$\langle\phi_i|\, P_{y,1}\,|\psi\rangle = \langle\phi_i|\,\frac{1}{2}(|\uparrow\rangle_e + |\downarrow\rangle_e\,|\downarrow\rangle + i\,|\downarrow\rangle_e\,|\uparrow\rangle - i\,|\uparrow\rangle_e\,|\downarrow\rangle) \tag{16}$$

Again for 1,2 we have

$$\langle\phi_{1,2}|\, P_{y,1}\,|\psi\rangle \quad = \quad \langle\phi_{1,2}|\,\frac{1}{2}(|\uparrow\rangle_e\,|\uparrow\rangle + i\,|\downarrow\rangle_e\,|\uparrow\rangle - i\,|\uparrow\rangle_e\,|\downarrow\rangle) \tag{17}$$

$$= \quad \frac{1}{4}(1 \pm \sqrt{2}sin(-\pi/4))) = 1/4(1 - \pm 1) \tag{18}$$

while for 3 and 4 they are $\frac{1}{4}(1 - (\pm 1)$.

When the King tells the Physicist which of the σ he measured, the physicist can tell him, knowing what the outcome was for the $|\phi_i\rangle$ he got, what the value was for the that choice of the σ.

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2) Assume that we have a Hamiltonian

$$H = \frac{1}{2}\left(\frac{p_1^2}{m_1^2} + \frac{p_2^2}{m_2} + k_1 x_1^2 + k_2 x_2^2 + 2\epsilon x_1 x_2\right) \tag{19}$$

a)What are the 4 eigenvalues $\pm i\omega_1,\ \pm\omega 2$ of the Hamiltonian equations for this Hamiltonian in terms of the constants $m_1, m_2, k_1, k_2, \epsilon$.

$$-i\omega x_1 = p_1/m1; \qquad -i\omega x_2 = p_2/m2; \tag{20}$$

$$-i\omega p_1 = -k_1 x_1 - \epsilon x_2; \qquad -i\omega p_2 = -k_2 x_2 - \epsilon x_1 \tag{21}$$

or

$$-m_1\omega^2 x_1 = -k_1 x_1 - \epsilon x_2 \tag{22}$$

$$-m_2\omega^2 x_2 = -k_2 x_2 - \epsilon x_1 \tag{23}$$

which has solutions only if

$$(m_1\omega^2 - k_1)(m_2\omega^2 - k_2) = \epsilon^2 \tag{24}$$

This is a quadratic real equation in ω^2 which means that ω^2 has two solutions. There will be equal solutions if $\omega^2 = 0$ is a solution (in which case $k_1 k_2 = \epsilon^2$), or if $\epsilon = 0$ and $k_1/m_1 = k_2/m_2$

$$\omega^2 = \frac{-(k_1+k_2) \pm \sqrt{(}((k_1+k_2)^2 - 4k_2k_2 + 4\epsilon^2)}{2(m_1+m_2)} \tag{25}$$

$$= \frac{-(k_1+k_2) \pm \sqrt{(k_1-k_2)^2 + 4\epsilon^2}}{2(m1+m2)} \tag{26}$$

Ie, having a non-zero interaction between the two oscillators ensures that one cannot have degenerate eigenvalues.

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b) Is there any condition on k_i, m_i, ϵ such that $\omega_1 = \omega 2$?

c) If $m_1 = m_2$, $k_1 = k_2$, is there any condition on ϵ such that the eigenvalues are not purely imaginary?

The square root can never be imaginary. However, if it is larger than $k1+k2$, then there is a solution for ω^2 which is negative. Ie, if $(k_1-k_2)^2+4\epsilon^2 > (k_1+k_2)^2$ or $\epsilon^2 > k_1k_2$ then ω^2 will have a solution which is negative, and the oscillator system will be unstable.

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d) What are the normalised (using the symplectic norm) eigenvectors if $m_1 = m_2$, $k_1 = k_2$ and $\epsilon \neq 0$?

$\omega^2 = (k \pm |\epsilon|)/m$.

There is a symmetry in the Hamiltonian in this case, where $x_1, p_1 \leftrightarrow x_2, p_2$. The two eigenvalues of this symmetry is that the system be symmetric or antisymmetric under interchange of x_1 or x_2. The antisymmetric solution has $(x_2, p_2) = (-x_1, p_1)$ and the equation then are

$$-i\omega p_1 = -(k-\epsilon)x_1 - i\omega x_1 = p_1/m \tag{27}$$

Thus

$$< x, p >= 2i(-ix_1^2)(\sqrt{(k-\epsilon)m} = 1 \tag{28}$$

$$x_1 = -x_2 = \frac{1}{(4(k-\epsilon)m)^{1/4}} \tag{29}$$

3. Consider the Hamiltonian $H = \frac{1}{2}(p^2 - x^2)$. What are the eigenvalues? of the Hamiltonian equations? Show that there are no purely imaginary eigenvalues.

$$-i\omega x = p \tag{30}$$
$$-i\omega p = x \tag{31}$$

or $\omega^2 = -1$ Thus $i\omega = \pm 1$.

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Find a positive norm, normalised values of the initial momentum and position. What is the time dependence of this mode. Show that its norm is independent of time explicitly.

Let us take the inital conditions $x_0 = 1/\sqrt{2}, p_0 = -i/\sqrt{2}$. (Note that there are many other possibilities.) This has positive norm

$$\frac{i}{2}(-i - i) = 1 \tag{32}$$

(Note there are many other possibilities. The only requirement is that x_0/p_0 not be real.

$$i(x_0^* p_0 - p_0^* x_0) = 0 \to i\left(\frac{x_0}{p_0}\right)^* = \frac{x_0}{p_0} \tag{33}$$

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With these intial conditions, the equations are

$$x = x_0 cosh(t) + p_0 sinh(t) = \frac{1}{\sqrt{2}}(cosh(t) - isinh(t)) \tag{34}$$
$$p = p_0 cosh(t) + x_0 sinh(t) = \frac{-i}{\sqrt{2}}(cosh(t) + isinh(t)) \tag{35}$$

Then

$$\begin{aligned} < x, x > &= \frac{i}{2}((cosh(t) + isinh(t))(-i)(cosh(t) + isinh(t)) - (i)(cosh(t) - isinh(t))(cosh(t) - isinh(t))) && (36)\\ &= cosh(t)^2 - sinh(t)^2 = 1 && (37) \end{aligned}$$

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Find the Annihilation and creation operator this mode, and show explicitly that they are independent of time.

————————————————- The solution of the equations for X, P is

$$X = X_0 cosh(t) + P_0 sinh(t) \tag{38}$$
$$P = P_0 cosh(t) + X_0 sinh(t) \tag{39}$$

and

$$\begin{aligned} A &= \ < x, X >= \frac{i}{\sqrt{2}}((cosh(t) + isinh(t))(P_0 cosh(t) + X_0 sinh(t)) - i(cosh(t) - isinh(t))(X_0 cosh(t) + P_0 s \\ &= (X_0 + iP_0)/\sqrt{2} \\ A^\dagger &= (X_0 - iP_0)/\sqrt{2} \end{aligned}$$

Which are clearly time independent.

Suppliment:

If we want to write the state $\Psi(x,t) = \langle x| |0\rangle$ where 0 is defined by $A|0\rangle = 0$ and $X(t)|x\rangle = x|x\rangle$ then the equation for $\Psi(x,t)$ is $< x, X > \Psi(x,t) = 0$ or

$$\frac{i}{\sqrt{2}}[(cosh(t) + isinh(t)(-i\partial_x))\Psi(x) + -i(cosh(t) - isinh(t))x\Psi(x,t) = 0 \quad (43)$$

$$\Psi(t,x) = exp\left(-\frac{cosh(t) - isinh(t)}{cosh(t) + i\sinh(t)}\frac{x^2}{2}\right) \quad (44)$$

$$= exp\left(-\frac{1 - isinh(2t)}{cosh(2t)}\frac{x^2}{2}\right) \quad (45)$$

This is the full time dependent Schroedinger solution, obtained without having to solve the messy time depedent Schroedinger equation.

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The following is a bonus question. The max mark on the assignment is for the first three question, but this one's mark can increase your mark (to over 100possibly)

4. Given a field $\phi(t,x)$ with a Lagrangian

$$L = \sum_n \left(\partial_t\phi(t,x_n) - v\frac{(\phi(t,x_{n+1} - \phi(t,x_{n-1})}{2\Delta x}\right)^2 - \left(\frac{(\phi(t,x_{n+1}) - \phi(t,x_{n-1})}{x_{n+1} - x_{n-1}}\right)^2 \quad (46)$$

where $x_0 = 0$, $x_N = L$, $\Delta x = x_n - x_{n-1}$ is independent of n, and $x_{n+N} \equiv x_n$. (periodic boundary conditions).

(This is the model for sound waves in a one dimensional lattice with unit sound velocity).

a) Find the conjugate momenta to $\phi(x_n)$ and the Hamiltonian for this problem.

The conjugate momentum is is the veriational derivative with respect to $\partial_t\phi(x_n)$ which in this case is

$$\pi(x_n) = 2(\left(\partial_t\phi(t,x_n) - v\frac{(\phi(t,x_{n+1} - \phi(t,x_{n-1})}{2\Delta x}\right) \quad (47)$$

(Note that I seem to have forgotten the standard $\frac{1}{2}$ before the Lagrangian so this explains the factor of 2 in front.

The Hamiltonian is

$$H = \sum_n \pi(x_n)\partial_t\phi(x_n) - L \tag{48}$$

$$= \sum_n \pi(x_n)(\frac{1}{2}\pi(x_n) + v\frac{(\phi(t,x_{n+1} - \phi(t,x_{n-1})}{2\Delta x} - \left(\frac{1}{4}\pi_(x_n)^2 - \left(\frac{(\phi(t,x_{n+1}) - \phi(t,x_{n-1})}{x_{n+1} - x_{n-1}}\right)^2\right) \tag{49}$$

$$= \frac{1}{4}\pi_(x_n)^2 - \left(\frac{(\phi(t,x_{n+1}) + \phi(t,x_{n-1})}{x_{n+1} - x_{n-1}}\right)^2 + \pi(x_n)v\frac{(\phi(t,x_{n+1} - \phi(t,x_{n-1})}{2\Delta x} \tag{50}$$

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b)Find the equations of motion for this field. If a mode for the field has the solution $\phi(t,x) \propto e^{-i(\omega t - kx)}$ what are the possible values of k and the relation between ω, k for a solution to the equations of motion.

$$\partial_t\phi(x_n) = \frac{1}{2}\pi(x_n) + v\frac{(\phi(t,x_{n+1} - \phi(t,x_{n-1})}{2\Delta x} \tag{51}$$

$$\partial_t\pi(x_n) = +\frac{\phi(x_{n+1} + \phi_(x_{n-1}) - 2\phi(x_n)}{\Delta(x)^2} - \frac{1}{2}v\frac{\pi(x_n+1) - \phi(x_n-1)}{2\Delta x} \tag{52}$$

Note that "n" in the Hamiltonian is a "dummy variable" since it is summed over. Thus you need to find all of the values of that dummy variable which have the value n.

If we choose $\phi_k(t,x_n) \propto e^{-i(\omega t - kx_n)}$, π_k would also have to have the same dependence, and the equations of motion become

$$-i\omega\phi_k e^{-i(\omega t - kx_n)} = (\frac{1}{2}pi_k + v\phi_k(\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x})e^{-i(\omega t - kx_n)} \tag{53}$$

$$-i\omega\pi_k e^{-i(\omega t - kx_n)} = -\phi_k\frac{e^{ik\Delta x} + e^{-ik\Delta x} - 2}{\Delta x^2}e^{-i\omega t - kx_n} - \frac{1}{2}v\pi_k\frac{e^{ik\Delta x} - e^{-ik\Delta x}}{2\Delta x}e^{-i\omega t - kx_n} \tag{54}$$

or

$$-i\omega\phi_k = (\frac{1}{2}pi_k + v\phi_k i\frac{\sin(k\Delta x)}{\Delta x} \tag{55}$$

$$-i\omega\pi_k = (\phi_k 2\frac{\cos(k\Delta x) - 1}{\Delta x} + iv\frac{1}{2}\pi_k\frac{\sin(k\Delta x)}{\Delta x} \tag{56}$$

The system is also periodic in n, so that $x_N + 1 = x_1$ which means that $e^{ik(N+1)\Delta x} = e^{ik\Delta x}$ or $Nk\Delta x = 2\pi r$ where r is some integer. Thus

$$k = 2\pi\frac{r}{Ni\Delta x} \tag{57}$$

Putting this into the above, we get

$$(\omega - v\frac{sin(2\pi\frac{r}{N})}{2\Delta x})^2 = \frac{cos(2\pi\frac{r}{N}) - 1}{\Delta x^2} \tag{58}$$

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c) What is the inner product for two of these solution-modes with different k and ω.

$$< \phi_k, \phi_k' >= \frac{i}{2}\sum_n \left(\phi_k^* \pi_k' - pi_k^* \phi_k') e^{i((\omega-\omega')t-(k-k')x_n)} \right) = 0 \tag{59}$$

since

$$\sum_{n=1}^N e^{i2\pi r - r' n/N} = \frac{e^{i2\pi(r-r')(N+1)/N} - e^{i2\pi(r-r')(1/N)}}{(1-e^{i2\pi(r-r')}} = 0 \tag{60}$$

Thus all solutions with different k, k' have zero norm. The two solutions with the same k but diffferent ω (one the negati of th other) are complex conjugates of each other, and the norm of complex conjugates with the solution are also zero.

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d) What is the norm of a mode for a given value of ω, k. What is the relation between the sign of the norm and that of ω if $v < \frac{2}{pi\Delta x}$, if $v < 1/2\Delta x$ and for $v > 1/2\Delta x$?

The exponential part is

$$\sum_n 1 = N \tag{61}$$

Thus the norm is

$$N\frac{i}{2}\phi_k^* \pi_k - \pi_k^* \phi_k \tag{62}$$

But $-i(\omega - v\frac{\sin(2\pi kr)}{\Delta x})\phi_k = \frac{1}{2}\pi_k$ so the norm is $(\omega - v\frac{\sin(2\pi kr)}{\Delta x})N|\phi_k|^2$. The equation of motion give

$$(\omega - v\frac{\sin(2\pi\frac{r}{N})}{\Delta x}) = \pm\sqrt{2\frac{cos(2\pi r/overN - 1}{\Delta x^2}} = \pm 2\frac{sin(2\pi\frac{r}{2N}}{\Delta x} \tag{63}$$

or, taking the positive root which would have positive norm

$$\omega_r = 2\frac{sin(2\pi\frac{r}{2N}}{\Delta x} + v2\frac{sin(2\pi\frac{r}{N}}{\Delta x} \tag{64}$$

which is negative if $v > 1$ for r near N.

Thus, if $v > 1$ we find that the the sign of ω can be the opposite to the sign of $\omega - v\frac{\sin(2\pi\frac{r}{N})}{\Delta x}))$. Ie, the sign of ω can be opposite the sign of the norm. Positive norm can be negative frequency, and vice versa. It is the norm that the determines the annihilation and creation operators since

$$\left[< \{\phi_k e^{-(i\omega t - kx_n)}\}, \{\Phi(x_n)\} >, < \{(\phi_k e^{-i(\omega t - kx_n)})^*\}, \{\Phi(x_n)\} > \right] =< \{\phi_k e^{-i\omega t - kx_n)}\}, \{\phi_k e^{-i\omega t - kx_n)}\} > \tag{65}$$

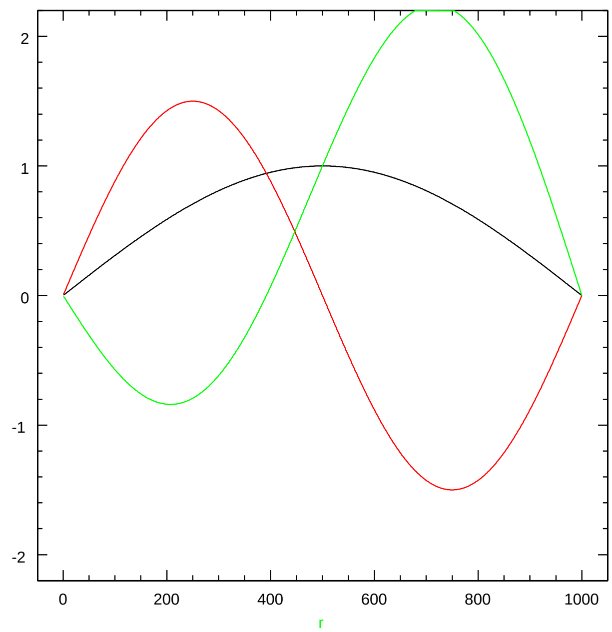

Figure 1: This is a plot of the norm, the frequency, and the velocity dependent term for the above as a function of r, with N=1000. The black curve is the norm, equal to $\Delta(x)(\omega + v\sin(2\pi\frac{r}{N})$ for $|\phi_k|^2 = 1$ with v=1.5 . The green curve is $\Delta x\omega$ while the red is the v dependent term. Note that for r less than about 400, ω is negative, while the norm is positive. Ie, the annihilation operator is associated with negative frequencies rather than positive in this regime.

which must be equal to +1 for Annihilation operators.

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Note that this would be even more interesting if we were to make the dispersion relation look like that for a BEC

$$L = \Delta x \sum_n \left(\partial_t \phi(t, x_n) - v \frac{(\phi(t, x_{n+1} - \phi(t, x_{n-1})}{2\Delta x} \right)^2 \tag{66}$$

$$- \left(\frac{(\phi(t, x_{n+1}) - \phi(t, x_{n-1})}{x_{n+1} - x_{n-1}} \right)^2 \tag{67}$$

$$- \kappa^2 \left(\frac{(\phi(t, x_{n+2}) - 2\phi(t, x_n) - \phi(t, x_{n-2})}{\Delta x^2} \right)^2 \tag{68}$$

which is the discrete Gross Piatevskii approximation to a flowing Bose Einsten Condensate where $\kappa << N\Delta x$ is the healing length of the BEC. But this is not the question I ask.