Physics 501-20
Assignment 1

1.) Hardy system: Given the state

$$|\Psi\rangle = \alpha\,|\uparrow\rangle\,|1\rangle + \beta\,|\downarrow\rangle\,(S\,|1\rangle + C\,|0\rangle) \tag{1}$$

where this is a unit vector with $|alpha|^2 + |\beta|^2 = 1$ and $|C|^2 + |S|^2 = 1$

i)Argue that we can always choose the coefficients are real and positive.

The normalised eigenstates remain normalised eigenstates even if one changes the phase of the eigenstate. Thus we can change the total phase of the wave function, and we can change the phase of each of the eigenvectors $|\uparrow\rangle\,,\ |\downarrow\rangle\,,\ |1\rangle\,,\ |0\rangle$. If C, S, α, β are complex, we can absorb the phase of C into $|1\rangle$,, S int $|0\rangle$, β into $|\downarrow\rangle$, and α into $|\uparrow\rangle$.

****

ii) Find the value of S that minimizes the probability of having the final value of "Y" be equal to +1.

Measurement of the projection onto $|\uparrow\rangle$ gives the second having $|1\rangle$. Measurement of $|1\rangle$ gives the first having $N(\alpha\,|\uparrow\rangle + \beta S \downarrow\rangle)$ as its normalised eigenvectory, with N being a normalisation factor. Meaurement of the projection onto $N(\alpha\,|\uparrow\rangle + \beta S)$ gives the second having the state

$$NM(\alpha^2 S\,|1\rangle + \beta^2 S(C\,|0\rangle + S\,|1\rangle) = NM(\alpha^2 S + \beta^2 S^2)\,|1\rangle + \beta^2 SC\,|0\rangle \tag{2}$$

as its state vector. The ratio of the probability that the second system has $|0\rangle$ as its state over that it has $|1\rangle$ (which is orthogonal to $|1\rangle$ is thus

$$\frac{P_0}{1-P_0} = \frac{\beta^4 S^2 C^2}{S^2(\alpha^2 + beta^2 S)^2} = \frac{\beta^4 C^2}{(\alpha^2+\beta^2 S)^2} = \frac{((1-\alpha^2)^2(1-S^2)}{(\alpha^2+\beta^2 S)} \tag{3}$$

Maximizing this S, we find $S = \frac{\alpha}{\sqrt{1+\alpha^2}}$ as the positive minimum.

****

iii) Given that value of S, is the largest value of the the ration of the eigenvalues $\lambda_1 \lambda_2$ where the two λ are the two eigenvalues of the reduced density matrix of particle 1 with $\lambda 1$ being the smallest of the eigenvalues.

The reduced density matrix for system2 is

$$\rho = \alpha^2\,|1\rangle\,\rangle 1| + \beta^2(S\,|1\rangle + C\,|0\rangle) * (S\,\rangle 1| + C\,\rangle 0|) \tag{4}$$

$$= (\alpha^2 + \beta^2 S^2)\,|1\rangle\,\rangle 1| + \beta^2 SC(|1\rangle\,\rangle 0| + |1\rangle\,\rangle 0|) + \beta^2 C^2\,|0\rangle\,\rangle 0|) \tag{5}$$

The eigenvalue equation is then

$$\lambda^2 - \lambda + ((\alpha^2 + \beta^2 S^2) beta^2 C^2 - \beta^2 S^2 C^2 \tag{6}$$

$$= \lambda^2 - \lambda + \alpha^2\beta^2 C^2 \tag{7}$$

$$= \frac{1}{2}(1 \pm \sqrt{1 - 4\alpha^2\beta^2 C^2} = \frac{1}{2}(1 \pm \sqrt{\frac{1+\alpha^4}{1+\alpha^2}} \tag{8}$$

Since $0 < \alpha < 1$, the eigenvalues go from $1/2, 1/2$ to $0, 1$. If α is small, the eigenvalues are $\alpha^2/4, 1 - alpha^2/4$ to lowest order in α. If α is near 1, the two eigenvalues are $1 - (1 - \alpha)/4, (1 - alpha)/4$ which are again near 0,1. The minimum difference between the two eigenvalues is when $a^2 = sqrt(2) - 1$, for which the two eigenvalues are approx (.09 and .91). Ie, this does not allow us to get anywhere near a maximal entanglement.
****

2) No Cloning:

Alice claims that she can duplicate a state, such that if the state is $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ and $|\phi\rangle = \alpha|\downarrow\rangle + \beta|\uparrow\rangle$ that she can start with a state $|\psi\rangle|\downarrow\rangle$ and create the state $|\psi\rangle|\phi\rangle$ without knowing what the coefficients α or β. Is she right and if not, why not, and if yes, how?

This is pretty straightforward. If one had a way to clone a state, one would start with a state like

$$|\Psi\rangle = (\alpha|\uparrow\rangle + \beta\downarrow)|0\rangle \tag{9}$$

Ie, one would start out with the system one wanted to clone to being in some fiducial state, here indicated by $|0\rangle$. After the cloning one would have

$$|\Psi\rangle' = (\alpha|\uparrow\rangle + \beta\downarrow)((\alpha|1\rangle + \beta|0\rangle) \tag{10}$$

But this is not a linear transformtion. It is clearly non-linear in α and β. But all transformations in QM are linear.

(Note that Wooters and Zurek got a PRL out of this observation).
****

3) Bell states and Quantum Teleportation:

i) Show that the four bell states over two two-level systems$i$ and j (basis vectors indicated by $|\uparrow_i\rangle$ and $|\downarrow_i\rangle$. i indicates which two level system is being referred to.) (Note that the state $|\uparrow_1, \uparrow_2\rangle$ say is equivalent to $|\uparrow_1\rangle|\uparrow_2\rangle$

$$|B_{ij0}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_i\downarrow_j\rangle - |\downarrow_i\uparrow_j\rangle) \tag{11}$$

$$|B_{ij1}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_i\downarrow_j\rangle + |\downarrow_i\uparrow_j\rangle) \tag{12}$$

$$|B_{ij2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_i\uparrow_j\rangle - |\downarrow_i\downarrow_j\rangle) \tag{13}$$

$$|B_{ij3}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_i\uparrow_j\rangle + |\downarrow_i\downarrow_j\rangle) \tag{14}$$

$$\tag{15}$$

are orthogonal to each other, are complete (ie, any state of a two-twelve system can be written in terms of them.)

$$\frac{1}{\sqrt{2}}(|B_{ij0}\rangle + |B_{ij1}\rangle) = |{\uparrow_i}{\downarrow_j}\rangle \tag{16}$$

$$\frac{1}{\sqrt{2}}(|B_{ij0}\rangle - |B_{ij1}\rangle) = |{\downarrow_i}{\uparrow_j}\rangle \tag{17}$$

$$\frac{1}{\sqrt{2}}(|B_{ij2}\rangle + |B_{ij3}\rangle) = |{\uparrow_i}{\uparrow_j}\rangle \tag{18}$$

$$\frac{1}{\sqrt{2}}(|B_{ij2}\rangle - |B_{ij3}\rangle) = |{\downarrow_i}{\downarrow_j}\rangle \tag{19}$$

The RHS are clearly a complet set of states in terms of which any other two-twolevel system's state can be written in terms of.

$|B_{ij0}\rangle$, and $|B_{ij1}\rangle$are clearly orthodonal to $|B_{ij2}\rangle$, and $|B_{ij3}\rangle$ since all the components are orthogonal.

Also $|B_{ij1}\rangle$ and $|B_{ij0}\rangle$ are orthogonal, since $\rangle{\uparrow_i}{\downarrow_j}|\,|{\uparrow_i}{\downarrow_j}\rangle = \rangle{\downarrow_i}{\uparrow_j}|\,|{\downarrow_i}{\uparrow_j}\rangle = 1$ and $\rangle{\uparrow_i}{\downarrow_j}|\,|{\downarrow_i}{\uparrow_j}\rangle = 0$

Each of the 4 terms$|{\uparrow_i}{\downarrow_j}\rangle$ are unit vectors and thus the sum of two of them (as all the B's are) would have a norm of 2, which is what the $\frac{1}{\sqrt{2}}$ compensates for.

****

Now interpret $|\uparrow\rangle$ as the $\sigma_z = 1$ eigenstate for the Pauli matrix, and $|\downarrow\rangle$ as the $\sigma_z = -1$ eigenstate of that same z Pauli matrix. system. The $|B\rangle_{ij0}$ and $|B\rangle_{ij1}$ are -1 eigenstates of the $Z_{ij1} = \sigma_{iz}\sigma_{jz}$ operator and the other two are +1 eigenstates of that same operator. Also $|B\rangle_{ij0}$ and B_{ij2} are -1 eigenstates of the operator which exchanges $|\uparrow_i\rangle$ with $|\downarrow_i\rangle$ and $|\uparrow_j\rangle$ with $|\downarrow_j\rangle$ (the $Z_{ij2} = \sigma_{ix}\sigma_{jx}$ operator) and the other two are the +1 eigenstates of this operator. Define $Z_{ijN} = \frac{1}{2}(Z_1+1)+(Z_2+1)$. It has eigenvalues or $\{0, 1, 2, 3\}$ and the eigenvectors are B_{ijN}.

Now, introduce a third system in an unknown arbitrary state$\alpha\,|\downarrow_3\rangle + \beta\,|\uparrow_3\rangle$, and assume that the initial state of particles 1 and 2 is $|B_{120}\rangle$

ii) One now measures the operator Z_{13N} What is the state of of second particle after this measurement for the four possible eigenvalues? Show that the probabilities each of the possible outputs are all equal. Ie, the outcome of the measurement of Z_{13N} give no hints as to the values of α and β.

The B_{13n} are the eigenvectors of Z_{13N}, We have

$$\begin{aligned}
\rangle B_{130}|\,|B\rangle\,120(\alpha\,|\downarrow_3\rangle + \beta\,|\uparrow_3\rangle) &= \frac{1}{2}\rangle\uparrow_1|\,\rangle\downarrow_3|\,(\alpha\,|\uparrow_1\rangle\,|\downarrow_2\rangle\,|\downarrow_3\rangle + \alpha\,|\downarrow_1\rangle\,|\uparrow_2\rangle\,|\downarrow_3\rangle) && (20)\\
&+\rangle\uparrow_1|\,\rangle\downarrow_3|\,(\beta\,|\uparrow_1\rangle\,|\downarrow_2\rangle\,|\uparrow_3\rangle + \beta\,|\downarrow_1\rangle\,|\uparrow_2\rangle\,|\uparrow_3\rangle) && (21)\\
&+\rangle\downarrow_1|\,\rangle\uparrow_3|\,(\alpha\,|\uparrow_1\rangle\,|\downarrow_2\rangle\,|\downarrow_3\rangle + \alpha\,|\downarrow_1\rangle\,|\uparrow_2\rangle\,|\downarrow_3\rangle) && (22)\\
&+\rangle\uparrow_1|\,\rangle\downarrow_3|\,(\beta\,|\uparrow_1\rangle\,|\downarrow_2\rangle\,|\uparrow_3\rangle + \beta\,|\downarrow_1\rangle\,|\uparrow_2\rangle\,|\uparrow_3\rangle) && (23)\\
&= \frac{1}{2}(\alpha\,|(\rangle\downarrow_2) + \beta\,|\uparrow_2\rangle) && (24)
\end{aligned}$$

The norm squared of this vector which is the probability of obtaining this vector, is $\frac{1}{4}$.

Similarly

$$\rangle B_{131}|\,|B_{120}\rangle\,(\alpha\,|{\downarrow_3}\rangle + \beta\,|{\uparrow_3}\rangle)) = \frac{1}{2}(\alpha\,|{\downarrow_2}\rangle - \beta\,|{\uparrow_2}\rangle) \tag{25}$$

$$\rangle B_{132}|\,|B_{120}\rangle\,(\alpha\,|{\downarrow_3}\rangle + \beta\,|{\uparrow_3}\rangle)) = \frac{1}{2}(\beta\,|{\downarrow_2}\rangle + \alpha\,|{\uparrow_2}\rangle) \tag{26}$$

$$\rangle B_{133}|\,|B_{120}\rangle\,(\alpha\,|{\downarrow_3}\rangle + \beta\,|{\uparrow_3}\rangle)) = \frac{1}{2}(\beta\,|{\downarrow_2}\rangle - \alpha\,|{\uparrow_2}\rangle) \tag{27}$$

The first term expression is exactly the state of the system 3 transfered to 2. The second is the state but with a minus sign in front of the $\uparrow_2$. Thus if one operates on this state with a unitary operator which switches the sign of the $\uparrow_2$ term, we again get the state of 3 transfered to 2. In the third, if we switch $|{\uparrow_2}\rangle$ with $|{\downarrow_2}\rangle$ we again get the switch we want, and for the fourth, if we switch and change the sign of the resultant $|{\uparrow_2}\rangle$ we get the state transfered to 2.

Thus, as long as we measure some operator on 1 and 3 whose eigenvectors are the Bell states, and whose eigenvalues differ for each of the Bell states and we transfer to the person at 2 which of the eigenvalues we got, the person at 2 can transform his state on his system with the appropriate transformation to result in the state of 3 having been exactly transfered to 2.

We could have set up the entangled state between 1 and 2 and then sent 2 far way, this procedure would transfer the unkown state of 3 exactly to that system 2. Of course after the measurement on 1 and 3, they will be left in one of the Bell states, and thus retain no memory whatsoever of what that intial state of 3 was. The only memory will be in system 2.

****

iii)Show that for each measured outcome there is a simple unitary transformation which converts the state of the second particle into $\alpha\,|0_2\rangle + \beta\,|0_3\rangle$. Ie, one has transformed the unknown state of the first particle 3 into the same state for particle 2. Note that this is true even if the second particle is over at Alpha Centauri and particles 1 and 3 are on earth.

Ie, a joint measurement on particles 1 and 3 which gives no information about that unknown state, and the classical transmission of the outcome of that measurement of that apparently useless measurement to someone who has system 2, allows someone at the location of 2 to make an exact copy of that unknown state of system 3, but at the expense of destroying the original state of system 3. Ie, this does not run afoul of the "No-cloning" theorem. There are never two copies of the initial state of system 3.

In each case the state of the system 2 is clearly related to the original. In the case of $|B_{130}\rangle$ is measured, for the joint system 1 and 3, the state of 2 is $\alpha\,|{\uparrow}\rangle + \beta\,|{\downarrow}\rangle$, which is exactly the intial state transfered to particle 2. Thus one has to operate on the system 2 with the identity matrix, which is a unitary matrix. In the case $|B_{131}\rangle$ is measured, it is $\alpha\,|{\uparrow}\rangle - \beta\,|{\downarrow}\rangle$. This can be transfored to the original state of particle 3 by a unitary transformation which changes the sign of $|\downarrow\rangle$, namely Σ_z. In caset B_2 was measured on particles 1 and 3, the final state of 2 is $\alpha \downarrow + \beta \uparrow$ which can be transformed to the required state by interchanging uparrow and downarrow – the unitary operator Σ_x. Finally if $|B_{133}\rangle$ was measured, then the operator is Σ_y. While this also multiplies the state by the pure phase $i = e^{i\pi/2}$ multiplication by a pure phase of course leaves the state the same.

Note that these measurements are done by measuring the Z_{13N} operator on the joint system of 1 and 3. If $Z_{13N} \to 0$ we have the state B_{130}. If $Z_{13N} \to 1$ the $|B_{131}\rangle$, if $Z_{13N} \to 2$ then $|B_{132}\rangle$ and if $Z_{13N} \to 3$ then $|B_{133}\rangle$

****