Physics 501-20
Assignment 1

1.) Consider the Hamiltonian

$$H = \frac{1}{2}(\frac{1}{m}P^2 + m(\Omega^2 + \epsilon\delta(t-1))Q^2) \tag{1}$$

Assume that at $t = 0$ the operators P, Q are given by $P = P_0, \quad Q = Q_0$.

Assume that the intial state of this quantum system is the lowest energy state at time t=0 (the ground state).

a)Solve the Heisenberg representation equations for the operators $P(t),\ Q(t)$ for this system at arbitrary times t.

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Assume $\hbar = 1$

$$\partial_t Q = -i[Q, H] = -\frac{i}{2m}[Q, P^2] = \frac{1}{m}P \tag{2}$$

$$\partial_t P = -i[P, H] = -\frac{i}{2}[P, m\Omega^2 Q^2 + \epsilon\delta(t-1)Q^2] = -(m\Omega^2 + 2\epsilon\delta(t))Q \tag{3}$$

For both t¡1 and t¿1 this is just the free Harmonic oscillator

$$Q = Q_0 cos(\Omega t) + P_0\frac{sin(\Omega t)}{\Omega m} \tag{4}$$

$$P = P_0 cos(\Omega t) - Q_0 \sin(\Omega t) m\Omega \tag{5}$$

$$\tag{6}$$

$t > 1$

$$Q = Q_1 cos(\Omega(t-1)) + P_1\frac{sin(\Omega(t-1)}{\Omega m} \tag{7}$$

$$P = P_1 cos(\Omega(t-1)) - Q_1 \sin(\Omega(t-1)) m\Omega \tag{8}$$

Where Q_1, P_1 are the values of the operators $Q,\ P$ just after t=1. Integrating from $1-\mu$ to $1+\mu$ where $\mu \to 0$, we have

$$Q_1 - (Q_0 cos(\Omega 1) + P_0\frac{sin(\Omega 1)}{m\Omega} = 0 \tag{9}$$

$$P_1 - (P_0 cos(\Omega 1) - Q_0 sin(\Omega 1) m\Omega = 2\epsilon Q_1 \tag{10}$$

or

$$Q_1 = Q_0 cos(\Omega 1) + P_0\frac{sin(\Omega 1)}{m\Omega} \tag{11}$$

$$P_1 = P_0 cos(\Omega 1) - (m\Omega + 2\epsilon)Q_1 \tag{12}$$

Thus after $t = 1$, we have

$$\begin{aligned} Q(t) &= Q_0 cos(\Omega t) + P_0\frac{sin(\Omega t)}{\Omega m} && (13)\\ P(t) &= P_0 cos(\Omega t) - Q_0 \sin(\Omega t) m\Omega && (14)\\ &\quad +2\epsilon(Q_0 cos(\Omega 1) + P_0\frac{sin(\Omega 1)}{m\Omega})cos(\Omega(t-1) && (15) \end{aligned}$$

With $\sqrt{m\Omega}Q_0 = \frac{1}{\sqrt{2}}(A_0 + A_0^\dagger)$ and $\frac{P_0}{\sqrt{m\Omega}} = \frac{i}{\sqrt{2}}(A_0 - A_0^\dagger)$

At $t = t_1 + \mu$ for $\mu \to 0$ we can write

$$P_1 == sqrt\frac{m\Omega}{2}(A_1^\dagger - A_1) \tag{16}$$

$$Q_1 = \frac{1}{\sqrt{2m\Omega}}(A_1^\dagger + A_1) \tag{17}$$

we find

$$A_1 = A_0 e^{-i\Omega 1}(1 + i\frac{\epsilon}{m\Omega}) + i\frac{\epsilon}{m\Omega}A_0^\dagger e^{i\Omega 1} \tag{18}$$

The energy after $t = 1$ is just

$$H_1 = \frac{\Omega}{2}(2A_1^\dagger A_1 + 1) \tag{19}$$

as usual for a Harmonic oscillator. Writing this in terms of A_0 we have

$$H_1 = (\frac{\Omega}{2} + \frac{epsilon^2}{\Omega m^2})(A_0^\dagger A_0 + A_0 A_0^\dagger) + \text{termsin}A_0^2\text{and}(A_0^\dagger)^2 \tag{20}$$

so

$$\rangle 0|\, H_1 \,|0\rangle = \frac{\Omega}{2} + \frac{epsilon^2}{\Omega m^2} \tag{21}$$

(since $A_0 |0\rangle = \rangle 0| A_0^\dagger = 0$)

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b)Define the Annihilation operators $A_0 = \frac{1}{sqrt2}(\sqrt{m}Q_0 + \frac{1}{\sqrt{m}}P_0)$. Show that $[A_0, A_0^\dagger] = 1$ and that $H(0) = \frac{1}{2}A_0 A_0^\dagger + A_0^\dagger A_0$. The minimum energy state $|0\rangle$ will therefor be given by

$$A_0 |0\rangle = 0 \tag{22}$$

Given the solution of the Heisenberg equations of motion, write the solutions for $P(t), Q(t)$ in terms of $A_0,\ \ A_0^\dagger$.

c) Find the expectation value of the energy H in the state $|0\rangle$ as a funtion of time except at $t = 1$.

d)Solve the Schroedinger equation for this problem with the same initial conditions, and explicitly find the expectation value of the energy as a function of time. (If necessary, solve this to lowest non-trivial order in ϵ. Note that the Heisenberg equations can be solved exactly to all orders in ϵ).

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The Schroedinger equation is

$$\partial_t\psi(x,t) = -iH|\psi\rangle = -i(-\frac{1}{2m}\partial_x^2\psi(x,t) + m\Omega^2x^2\psi(x,t) + \epsilon\delta(t-1) \tag{23}$$

The ground state has energy $\Omega/2$ and is Gaussian in x or

$$\psi(x,t) = e^{-i\Omega t/2} e^{\frac{-x^2}{2m\Omega}} \tag{24}$$

. At t=1 we have to get the solution past the delta function. This is not trivial, but there is a trick. at the delta function, the delta function dominates the evolution, so we can write the equation as

$$\partial_t \psi(x,t) = -i\epsilon x^2 \delta(t-1)\psi(x,t) \tag{25}$$

dividing by $\psi(x,t)$ we have

$$\partial_t ln(\psi(t,x)) = -i\epsilon x^2 \delta(t-1) \tag{26}$$

or

$$ln(\psi(x,1+\mu) - ln(\psi(x,1-\mu)) = -i\epsilon x^2 \tag{27}$$

or

$$\psi(x,1+\mu) = e^{-i\Omega 1} e^{-x^2(\frac{1}{\sqrt{2m\Omega}}+i\epsilon)} \tag{28}$$

This is not an energy eigenstate, so after $t=1$ the solution in time is very complex. However, since the energy is conserved, we can calculate the expectation value of the energy at time $t=1+\mu$. This of course gives us the same answer as for the Heisenberg representation.

Which is easier?
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The answer is up to you. If one simply wants to calculate the energy expectation value this one is easier. If one wants to calculate the full time dependence, the Heisenberg represenation is much easier, unless the state happens to be an energy eigenstate.

Note the state of the system after the t=1 is called a squeezed state, which has become a very powerful took in quantum optics in increasing the sensitivity of certain detectors beyond what one might naively call the quantum limit. We will look at this later in the course.