Physics 501-20

Norm

We begin with a simple Harmonic Oscillator with the Lagrangian

$$\mathcal{L} = \frac{1}{2}m(\partial_t x)^2 - kx^2 \tag{1}$$

From which we get the conjugate momentum

$$p = \frac{\delta \mathcal{L}}{\delta \partial_t x} = m\partial_t x \tag{2}$$

and the Hamiltonian

$$H = p\partial_t x - \mathcal{L} \tag{3}$$

where we write $\partial_t x$ in terms of p.

$$H = p(\frac{p}{m} - \frac{1}{2}(m\left(\frac{p}{m}\right)^2 - kx^2) \tag{4}$$

$$= \frac{1}{2}(\frac{p^2}{m} + kx^2) \tag{5}$$

The Hamiltonian action is

$$\int (p\partial_t x - H)dt = p\partial_t x - \frac{1}{2}(\frac{p^2}{m} + kx^2) \tag{6}$$

To quantize this system, we replace the finctional attributes of the system, p and x by operator attributes, where the operators are linear transformations of the vectors in the Hilbert space (represented by $|\psi\rangle$). $X,\ P$ which represent the "position" (or more generally the configuration variable) and its conjugate momentum, the attributes of this simple system, where

$$[X, P] = i\hbar I \tag{7}$$

where I is the identity matrix on the Hilbert space.

The states $|\phi\rangle$ are members of this Hilbert space (a vector space with a positive definite complex norm) and the probability of a n attribute A (represented by an Hermitian operator$B$) being measured to have a value b is given by the normalised eigenvector of B with eigenvalue b

$$B\,|b\rangle = b\,|b\rangle \tag{8}$$

$$\langle b|\,|b\rangle = 1 \tag{9}$$

(if B is an operator with a continuous spectrum (set of eigenvalues a) the eigenvectors are not normalisable in this above sense, and instead one demands that

$$\langle b|\,|b\rangle' = \delta(b, b') \tag{10}$$

where $\delta(b,b')$ is the Dirac delta "function" (actually a distributions) such that for any function $f(a)$ we have

$$\int \delta(b,b')f(b')da' = f(b) \tag{11}$$

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This continuum normalisation replaces the normalisation above if the sets of eigenvectors are actually normalisable (ie, have a finite Hilbert space norm).

Then the probability of measuring B to have value b is

$$\text{Prob}(b) = |\langle b| |\psi\rangle|^2 \tag{12}$$

The quantity $\langle b| |\psi\rangle$ is called the amplitude for the value b if the state of the system is $|\psi\rangle$.

Often in elementary quantum systems, the Hilbert space is taken as the functions over the eigenvalues of some operator B, and those functions have the norm of the integral (or sum) of the functions over the possible eigenvalues of A.

Inorder that the attributes have real eigenvalues, we require that

$$B^\dagger = B \tag{13}$$

Of course one is often interested in the time developement of a physical system. There are two popular ways of representing that time development. One if the Heisenberg representation. This regards the operators (just like the values of attribute in the classical syste) as time dependent. The state is a constant. The operators obey the operator equation

$$i\hbar\partial_t B = [B,H] \tag{14}$$

Note that this equation has no state, and these equations are independent of state. The operators carry the time development of the system.

The other representation is the Schroedinger representation. In this case it is the state that carries the time dependence.

$$i\partial_t |\psi,t\rangle = H|\psi,t\rangle \tag{15}$$

We note that the two representations carry the same amplitudes. Consider the Unitary operator which obeys

$$i\hbar\partial_t U(t,t_0) = HU(t,t0) \tag{16}$$

$$U(t_0,t_0 = I \tag{17}$$

If H is time independent, this has the (formal) solution

$$U(t,t_0) = e^{-i\int_{t_0}^{t} Hdt/hbar} = e^{-iH(t-t+0)/hbar} \tag{18}$$

where that exponential is defined in terms of the power series representation of the exponential.

There are some delicate features of the matrix in the case that the Hamiltonian is unbounded (eigenvalues of arbitrarily large values), since the radius of convergence of the power series may vanish for the operator on a generic vector in the Hilbert space (the vector may have components with arbitrarily large eigenvalues). But we will ignore this here.

We note that the amplitudes are equivalent for the two representations. Given to states, $|\phi\rangle, |\phi'\rangle$, the expectation value of some operator is using the Unitary operator, we find that

$$B(t) = U(t,t_0)^\dagger BU(t,t_0) \tag{19}$$

is a solution to the equations of motion

$$i\hbar\partial_t(U(t,t_0)^\dagger BU(t,t_0)) = (-i\hbar\partial_t U)^\dagger BU + i\hbar + U^\dagger B(i\hbar\partial_t U) \tag{20}$$
$$= U^\dagger[B,H]U = [U^\dagger BU, U^\dagger HU] = [B(t),H] \tag{21}$$

Now, for our Hamonic oscillator, we know how to solve the classical equations.

$$\partial_t x = \frac{p}{m} \tag{22}$$
$$\partial_p = -kx \tag{23}$$

which gives the linear equation for x of

$$\partial_t^2 x - \frac{k}{m}x = 0 \tag{24}$$

which has as solution

$$x(t) = x_0 cos(\omega(t-t_0)) - (p_0/(m\omega))sin(\omega(t-t_0)) \tag{25}$$
$$p(t) = p_0 cos(\omega(t-t_0) + x_0 m\omega sin(\omega(t-t_0) \tag{26}$$

where $\omega = \sqrt{k/m}$ and $x_0,\ p_0$ are the values of the $x,\ p$ at time t_0. .

The quantum equations for $X,\ P$ are identical, as are the solutions

$$\hbar\partial_t X = [X,H] = i\hbar\frac{P}{m}; \qquad i\hbar\partial_t P = [P,H] = -i\hbar kX \tag{27}$$

with solutions

$$X(t) = X(t_0)cos(\omega(t-t_0)) + P(t_0)/(m\omega)sin(\omega(t-t_0)) \tag{28}$$
$$P(t) = P_0(t_0)cos(\omega(t-t_0)) - X(t_0)m\omega sin(\omega(t-t_0)) \tag{29}$$

This solution is so easy because the equations of motion are linear, and so any sum of solutions is also a solution to the equations of motion. The Heisenberg representation is far more difficult for systems whose attributes obey non-linear equations of motion.

The Schroedinger equations are

$$i\partial_t U(t,t_0)\left|\phi,t_0\right\rangle = HU(t,t_0)\left|\phi,t_0\right\rangle \tag{30}$$
$$\tag{31}$$

Thus in the Heisenberg representaiton

$$\begin{aligned}\left\langle\phi\right| U^\dagger B(t_0) U \left|\phi'\right\rangle &= (U\left|\phi\right\rangle)^\dagger B(t_0) U \left|\phi\right\rangle' &(32)\\ &= \left\langle\phi,t\right| B(t_0) \left|\phi,t\right\rangle &(33)\end{aligned}$$

Modes

For a linear system, a mode is solution of the equations of motion for a linear set of classical equations. For example, for the equations for the Harmonic oscillator the set of solutions, for the various choices of $x(t_0), \quad, p(t_0)$ are all modes. We will define a norm for these modes. Note that this is NOT a norm on some Hilbert space, but is norm on the solutions of these linear equations

Given two complex solutions $x(t), \quad p(t)$ and $x'(t), \quad p(t)$, I will define the norm

$$< \{x(t),p(t)\},\{x'(t),p'(t)\} >= i(x(t)^*p'(t) - p^*(t)x'(t)) \tag{34}$$

This clearly means that the solutions must be complex solutions. But since the equations are linear, if $x(t), p(t)$ is a solution, so it $ix(t), ip(t)$ (or any other complex constant times the solution).

In order not to drag along too many symbols, I will write

$$< x, x' > \equiv < \{x(t),p(t)\},\{x'(t),p'(t)\} > \tag{35}$$

Ie, although $< x, x' >$ is written solely in terms of the configuration variable, it is taken to mean the full solution in phase spece, not just configuration space.

I will also take $\hbar = 1$ in order to simplify the equations. It can be replaced by using dimensional analysis.

We note that we can define a norm for a solution by $< x(t), x(t) >$. This norm is zero is $x(t), p(t)$ is a real solution, since the two terms cancel. Also $< x^*, x^* >= - < x, x >$. Ie, for each positive norm solution there is a negatie norm solution which is the complex conjugat of the positive norm solution.

This norm has the property that $x(t), x'(t)$ are two complex solutions, then

$$\begin{aligned}\partial_t < x(t), x(t') > &= i(\partial_t x(t)^* p'(t) + x(t)^*\partial_t p'(t) - \partial_t p(t)^* x(t) - p(t)^*\partial_t x'(t)) &(36)\\ &= i((p(t)^*/m)p'(t) - x(t)^* k x'(t) + k x(t) x'(t) - p(t)^* p(t)/m) &(37)\end{aligned}$$

Ie, the norm of two solutions is a constant in time. It does not matter when the norm is evaluated.

These features of the norm are true no matter what the Hamiltonian for the linear system is, as long as it is purely quadratic in the variables.

We note that we could take one of our soltutions to the the operators X and P, and the other to be one of the modes. Then $< x, X >$ will have all of the above characteristics. In this case of course the norm is an operator, not a number.

Let us choose some mode whose norm is positive with the value 1 (unit norm). We can take the inner product between that mode and the Heisenberg operator solution of the equations. Then we can define the operator

$$A_{x(t),p(t)} =< x(t), X(t) >= i(x(t)^*P(t) - p(t)^*X(t)). \tag{38}$$

However by the same argument as for two modes, this operator will be time independent. But it will not be a Hermitian operator. In fact

$$A^\dagger_{x(t),p(t)} = -i(x(t)P(t) - p(t)X(t)) = - < x^*(t), X(t) > \tag{39}$$

Also, since $[X(t), P(t)] = i\hbar I$ we find that

$$[A, A^\dagger] \;=\; i(-i)(-x^*(t)p(t)[P(t), X(t)] - p^*(t)x(t)[X, P]) \tag{40}$$

$$= \; i(x^*p - p^*x)I =< x, x > I = I \tag{41}$$

Furthermore $X = Ax(t) + A^\dagger x(t)^*$.

Now $A^\dagger A$ is Hermitian, and is positive If $|\phi'\rangle = A|\phi\rangle$ then $0 < \langle\phi|' |\phi\rangle' = \langle\phi| A^\dagger A |\phi\rangle$. The expectation value of $A^\dagger A$ in any state is positive, since it is just the norm of the state $A|\psi\rangle$, which is positive, and which makes all eigenvalues of that operator positive. But by the commutation relations of A and $A^\dagger$ we have

$$[A, A^\dagger A] = A \tag{42}$$

$$A^\dagger AA|n\rangle = A(n-1)|n\rangle = n - 1A|n\rangle \tag{43}$$

where n is the eigenvalue of $A^\dagger A$. This means that the operator !A! reduces the eigenvalue of $A^\dagger A$ by 1. If you keep doing this, either the eigenvalue has to become negative (which it cannot do) or A has to annihilate the state (the eigenvalue of $A^\dagger A$ becomes 0). Thus n must be an integer, its smallest value is 0.

Note that this is true no matter what the mode is.Each mode gives you its own $|0\rangle$ state, and in general they are not the same as each other.

If you choose the mode so that $\partial_t x(t) = -i\omega x;\;\; \partial_t p = -i\omega p$ then that mode is called the diagonalisation of the Hamiltonian mode. If you choose the mode, so that this is true at some spacific time, and ω is real, then using that mode, (or its complex conjugate if it has a negative norm) then that gives you the Hamiltonian diagonalisation mode, and the "number operators" is proportional to the Hamiltonian at that time.

In our above example, if ω is positive, then the norm is positive.

However, if we choose for example

$$H = \frac{1}{2}(p^2 - bpx + cx^2) \tag{44}$$

then

$$-i\omega x = p - bx; \qquad -i\omega p = -(cx - bp) \tag{45}$$

and $\omega^2 = c - b^2$ for which we have real ω only if $c > b^2$.

One can also have an "unstable" Hamiltonian– for example with $k < 0$ in our model. One can certainly quantize this in the same way. One cannot diagonalise the Hamiltonian, so that $AA^\dagger$ for any mode is not related to the Hamiltonian, but that does not detract from these modes defining a zero state and regarding $A^\dagger A$ as a sort of number operator.

If H is a function of time, Then, even if the mode diagonalizes the Hamiltonian at some time t, it will not do so in general at any other time.

Multiple interacting oscillators

Let us take a whole bunch of oscillators and couple them to each other linearly. We can write the Hamiltonian as

$$H = \frac{1}{2}\sum_{ij}(\mu_{ij}p_ip_j + 2b_{ij}p_ix_j + c_{ij}x_ix_j) \tag{46}$$

with each of the matrices being symmetric. We have the equations of motion as

$$\partial_t x_i = \sum_j(\mu_{ij}p_j + b_{ij}x_j) \tag{47}$$

$$\partial_t p_i = -\sum_j(b_{ij}p_j + c_{ij}x_j) \tag{48}$$

The norm for any pair of solutions of these equations is

$$< \{x_i, p_i\}, \{x'_i, p'_i\} > \equiv < x_i, x'_i > = i\sum_j(x_j^* p'_j - p_j^* x'_j) \tag{49}$$

Again, if both $x, \;\; x'$ are solutions to the above equations then

$$\partial_t < x_i, x'_i > = 0 \tag{50}$$

One can solve the operator equations to get

$$A_{x(t)} = < x_i(t), X_i(t) >; \quad A^\dagger = < x_i(t)^*, X_i(t) > \tag{51}$$

and

$$[A_{x_i}, A^\dagger] = < x_i(t), x'_i(t) > \tag{52}$$

$$[A_{x_i}, A_{x'_i}] = < x_i(t), x_i'^* > \tag{53}$$

Again, any positive norm set of solutions can be used to define a set of annihilation operators, and by normalising the modes, one can get

$$[A_\alpha, A_\beta^\dagger] = \delta_{\alpha,\beta} I \tag{54}$$

Again one can define a 0 state such that $A_\alpha \left|0\right\rangle = 0$

This is true for **any** set of orthonormal mode solutions. Once again, one can define a Hamiltonian diagonalisation by replacing ∂_t by $-i\omega$ in the equations of motion. (Hamiltonian diagonalisation). There will in general be 2N values of ω, half of them positive and the other half negative (and the modes being the complex conjugate of the positive ones)