Physics 501
Midterm Exam
Feb 26 2020

This exam is 1hr 15 min (75 min) in length. It is closed book.

On your answer sheet, please include a statement that you have not and will not receive help in answering any of the question from any other person, and that you have not and will not read any material, whether in books or online to help you answer these questions, and sign that declaration.

This exam has 5 questions, all of equal value.

When the exam has been marked, and handed back to you, you will have the option of redoing any of the questions on which you did not get full marks and handing the result in for remarking. As your midterm mark you will get the average of the mark you received on the midterm writing itself and the mark you get on the redo of the question, but in no case will you get less than that the mark you received on the midterm marking itself. This "re-do" will be due 1 week after the results of the marking of the original midterm are sent back to you, with no extention of that time. You may discuss this redo with others, and use any notes, text-books, etc. in doing so, except you may not simply copy someone else's solution(s).

1) Consider the sigma matrix

$$\sigma_\theta = cos(theta)\sigma_z + sin(\theta)\sigma_x \tag{1}$$

and the Hamiltonian for this system of

$$H = 0 \tag{2}$$

At 9AM σ_z was measured and found to have a value of +1. At 11AM σ_x was measured and found to have a value of +1. At 10AM σ_θ was measured . What is the probability that the value of +1 was found this measurement,as a function of θ given the above conditions.

Simplest:

$$P(s_\theta = 1) = \frac{|\langle x=1| |\theta=1\rangle \langle\theta=1| |z=1\rangle|^2}{|\langle x=1| |\theta=1\rangle \langle\theta=1| |z=1\rangle|^2 + |\langle x=1| |\theta=-1\rangle \langle\theta=-1| |z=1\rangle|^2} \tag{3}$$

Now if $|z=1\rangle \equiv |\uparrow\rangle, \quad |z=-1\rangle \equiv |downarrow\rangle$ then $|x=1\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ and $|\theta=1\rangle = cos(\frac{\theta}{2})|\uparrow\rangle + \sin(\frac{\theta}{2})|\downarrow\rangle$ and $|\theta=-11\rangle = -sin(\frac{\theta}{2})|\uparrow\rangle + \sin(\frac{\theta}{2})|\downarrow\rangle$

$$|\langle x=1| |\theta=1\rangle \langle\theta=1| |z=1\rangle|^2 = \frac{1}{2}(cos(\frac{\theta}{2}) + sin(\frac{\theta}{2}))^2(cos(\frac{\theta}{2})^2 \tag{4}$$

$$|\langle x=1| |\theta=-1\rangle \langle\theta=-1| |z=1\rangle|^2 = \frac{1}{2}(-sin(\frac{\theta}{2} + cos(\frac{\theta}{2})^2(sin(\frac{\theta}{2})^2 \tag{5}$$

and

$$\text{Prob} = \frac{(1+sin(\theta))cos^2(\frac{\theta}{2})}{(1+sin(\theta))cos^2(\frac{\theta}{2}) + (1-sin(\theta))sin^2(\frac{\theta}{2})} = \frac{(1+sin(\theta))cos^2(\frac{\theta}{2}}{(1\frac{1}{2}sin(2\theta))} \tag{6}$$

Alternatively:

$$\text{Probability} = \frac{\text{Prob}(\theta = 1 \mid z = 1)\text{Prob}(x = 1|\theta = 1)}{\text{Prob}(\theta = 1|z = 1)\text{Prob}(x = 1|\theta = 1) + \text{Prob}(\theta = -1|z = 1)\text{Prob}(x = 1|\theta = -1)} \tag{7}$$

3 marks for getting the concept right, 2 for getting the calculation.
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2. Consider the free particle Hamiltonian

$$H = \frac{1}{2}p^2 \tag{8}$$

Consider the complex mode solution

$$x = 1 - it \tag{9}$$

a)What is p for this mode?

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$$\partial_t x = \partial_p(H) = p \tag{10}$$

$$p = -i \tag{11}$$

(1 mark)
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b)What is the norm for this mode?

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$$\text{Norm} = \frac{i}{2}(x^* p - p^* x) = \frac{i}{2}(1 + it)(-i) - (i)(1 - it)) = 1 \tag{12}$$

2 mark
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c)What is the annihilation operator corresponding to this solution in terms of the momentum and position operators at time $t = 0$ and what is the annihilation operator in terms of the momentum and position operators at time t?

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The solution for X and P is

$$X(t) = X(0) + P(0)t \tag{13}$$

$$P(t) = P(0) \tag{14}$$

Then

$$A =< x, X >= \frac{i}{2}(x^* P(t) - p^* X(t)) = \frac{i}{2}((1 + it)(P(t)) - (i)X(t)) = (P(0) - iX(0)) \tag{15}$$

2 marks
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3) Give the argument from Hardy's chain argument, that classical Mechanics cannot mimic the results from a quantum system.a Go into some detail.

The key feature is that one has a quantum state for two particles and operators A and C for one and B and D for the other, all of which have eigenvalues of ± 1. The state is such that always if A is measured to have value +1 then B if is measured it always has value +1. If B is measured to have value +1 then always if C is also measured it has value +1. If C is measured to have value +1 and D is measured it always have value +1.

But if A is measured to have value +1 then if D is measured it rarely have value +1. The less entangled the state the rarer it is that in the last case D has value +1.

This seems to violate logic

$$A = 1 \rightarrow b = 1 \rightarrow C = 1 \rightarrow D = 1 \tag{16}$$

where $\rightarrow$ means "implies that". But

$$A = 1 \rightarrow D = -1 \tag{17}$$

almost always.

$$|\psi\rangle >= sin(\theta) |\uparrow\uparrow\rangle + cos(\theta) |\downarrow\rangle_1 (sin(\phi) |\uparrow\rangle_2 + cos(\phi) |\downarrow\rangle_2) \tag{18}$$

Lets take $sin(\phi) = tan(\theta) << 1$. The first state$(|...\rangle_1$ is of the first particle and the second $|...\rangle_2$ is of the second. Then we take $|A = 1\rangle = |\uparrow\rangle_1$ which gives

$$\begin{aligned}
|B = 1\rangle &= N(\langle\uparrow|_1 |\psi\rangle >= |\uparrow\rangle_2 && (19)\\
|C = 1\rangle &= N \langle\uparrow|_2 |\psi\rangle = \frac{(|\uparrow\rangle_1 + |\downarrow\rangle_1)}{\sqrt{(2}} && (20)\\
|D = 1\rangle &= N(\langle\uparrow| + \langle\downarrow|) |\psi\rangle = \frac{2sin(\theta) |\uparrow\rangle_2 + \sqrt{(1 - 2sin(\theta)^2} |\downarrow\rangle_2}{\sqrt{1 + sin(\theta)^2}} && (21)\\
&\approx 2\theta |\uparrow\rangle_2 + |\downarrow\rangle_2 && (22)
\end{aligned}$$

from which the probability that if A=1 then D=1 is approximately $4\theta^2$, which is very small by assumption.

This Hardy chain means that one cannot imagine that any attribute has a value even if it is not measured. If one did, then the logical chain should be valid, and it is not.

5 marks alltogetehr. 1 for setting up the problem, 2 for the implication series. 1 for the conclusion and 1 for the mathematics. (The above mathematics is only one of the possibilities. There are a number of different assumptions one could make. I do not mind them just sketching the argument)

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4)Show that the unitary matrix

$$U = e^{i(\alpha P + \beta X)} \tag{23}$$

is a translation operator on both X and P, where P and X are the usual momentum and position operators. What linear combination of the operators X and P does this operator not change at all?

$\alpha P + \beta X$ communtes with U, and thus

$$U^\dagger(\alpha P + \beta X)U = U^\dagger U(\alpha P + \beta X) = (\alpha P + \beta X) \tag{24}$$

Thus $\alpha P + \beta X$ is left alone by the unitary transformation.

$$[(\alpha P + \beta X), P] = i\beta \tag{25}$$

so

$$[U, P] = \sum_r [\frac{(i(\alpha P + \beta X)^r}{r!}, P] = \sum_r \frac{ri\beta(i)(i(\alpha P + \beta X)^{r-1}}{r!} = -\beta \sum_r \frac{(i(\alpha P + \beta X)^{r-1}}{(r-1)!} = -\beta U \tag{26}$$

So

$$U^\dagger P U = U^\dagger U P + \beta = P + \beta \tag{27}$$

Similarly

$$U^\dagger X U = X - \alpha \tag{28}$$

so the operator translates both X and P by β an $-\alpha$

Alternatively, one could use the Campbell Baker Haussdorf relation that

$$e^{i(\alpha P + \beta X)} = e^{i\alpha P} e^{i\beta X} e^{\alpha\beta[P,X]} \tag{29}$$

$$= e^{i\beta X} e^{i\alpha P} e^{\alpha\beta[X,P]} \tag{30}$$

$$= e^{i\alpha X/2} e^{i\beta P} e^{i\alpha X/2} = e^{i\beta P/2} e^{i\alpha X} e^{i\beta P/2} \tag{31}$$

and use those to show the above.

Finally, one can formally note that if we define $Q = \alpha P + \beta X$, then $[X, Q] = i\alpha$, so we can represent X as $i\alpha \frac{\partial}{\partial Q}$ so that $X e^{-iQ}\phi(Q) = i\alpha \frac{\partial e^{-iQ}\phi(Q)}{\partial Q} = (\alpha + U^\dagger X U)U^\dagger \phi$ or $U^\dagger X U = X - \alpha$.

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5)Consider the Lagrangian for a time dependent oscillator of

$$L = \frac{1}{2} t^2 \left((\partial_t X)^2 - X^2 \right) \tag{32}$$

a) What is the (time dependent) Hamiltonian for this problem.

$$P = \frac{\partial L}{\partial(\partial_t X)} = t^2 \partial_t X \tag{33}$$

$$\tag{34}$$

$$\partial_t X = \frac{P}{t^2} \tag{35}$$

$$H = P\partial_t X - L = \frac{1}{2}(\frac{P^2}{t^2} + t^2 X^2) \tag{36}$$

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b) Doing one round of the assymptotic adiabatic transformation, what are $\hat{X}$, $\hat{P}$, τ in terms of $X,\ P,\ t$? What is $\hat{H}$ in terms of $\hat{X}$, $\hat{P}$, τ?

$$\begin{aligned} \tilde{P} &= P/t \\ \tilde{X} &= tX \\ S &= \int dt(P\partial_t X - H) = \int dt(\tilde{P}t\partial_t(\frac{\tilde{X}}{t}) - \frac{1}{2}(\tilde{P}^2 + \tilde{X}^2) \end{aligned}$$

$$= \int dt(\tilde{P}\tilde{X} - \frac{\tilde{P}\tilde{X}}{t} - \frac{1}{2}(\tilde{P}^2 + \tilde{X}^2))$$

$$= \int dt(\tilde{P}\partial_t\tilde{X} - \frac{1}{2}(\tilde{P}^2 + \tilde{X}^2 + \frac{2}{t}\tilde{P}\tilde{X}) = \int dt(\tilde{P}\partial_t\tilde{X} - \frac{1}{2}((\tilde{P} + \frac{\tilde{X}}{t})^2$$

Setting $\tilde{X} = \hat{X}$ and $\tilde{P} = \hat{P} - \frac{\hat{X}}{t}$

$$S = \int dt(\hat{P} - \frac{\hat{X}}{t}\partial_t\hat{X} - \frac{1}{2}(\hat{P}^2 + \hat{X}^2(1 - \frac{1}{t^2})) \tag{43}$$

Now,

$$\int dt(-\frac{\hat{X}}{t}\partial_t\hat{X}) = \int dt(-\frac{1}{2t}\partial_t(\hat{X}^2) = \int dt(-\partial_t(\frac{\hat{X}^2}{2t}) - \frac{\hat{X}^2}{2t^2} \tag{44}$$

So

$$S = \int dt(\hat{P}\partial_t\hat{X} - \frac{1}{2}(\hat{P}^2 + \hat{X}^2) \tag{45}$$

Thus

$$\hat{H} = \frac{1}{2}(\hat{P}^2 + \hat{X}^2) \tag{46}$$

which is a time independent Hamiltonian, and is trivially solved.

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c) Using the annihilation operator expressed in terms of $X(t)$, $P(t)$, and the representation of $X(t)\psi(t,x) = x\psi(t,x)$, $P(t)\psi(t,x) = -i\partial_x\pi(t,x)$, find the function $\psi(t,x)$ which is the $||\rangle\, 0 >$ state for the above annihilation operator in part b). [This is the Schroedinger representation of that state, which is, or course, time dependent.)

AArgh. this was supposed to be part d) of question 1 which I got rid of, except instead it migrated. Ignore this for the marking. It makes no sense in this context.

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Marking: 1 for part a), 4 for part b).