Physics 501-20

Concepts: 1)Coherent state

Consider an annihilation operator A. The vacuum state is $A|0\rangle = 0$. Now define a new state $|a\rangle$ such that $A|a\rangle = a|a\rangle$. Ie it is an eigenstate of A. This must a some function of $f(A^\dagger)|0\rangle$.

$$Af(A^\dagger)|0\rangle = af(A^\dagger)|0\rangle \tag{1}$$

But $[A, A^\dagger] = 1$ and so one could formaly represent A as $\frac{A=\partial}{\partial A^\dagger}$ and thus

$$Af(A^\dagger)|0\rangle = \frac{\partial}{\partial A^\dagger}f(A^\dagger)|0\rangle = af(A^\dagger)|0\rangle \tag{2}$$

or $f(A^\dagger) = e^{aA^\dagger}$. Alternatively,

$$A, {A^\dagger}^N = \sum_{r=0}^{N-1} A^{\dagger\ r}[A, A^\dagger]A^{\dagger\ (N-1-r)} = NA^{\dagger\ N} \tag{3}$$

so $\sum_{N=0}^\infty \frac{d^N f}{dA^{\dagger\ N}}(0)\frac{A^{\dagger N}}{N!}$ is the Taylor series expansion of $f(A^\dagger)$, and $[A, f(A^\dagger)] = \frac{d^N f}{dA^{\dagger\ N}}(0)\frac{A^{\dagger(N-1)}}{(N-1)!}$ and thus $Af(A^\dagger)|0\rangle = af(A^\dagger)|0\rangle$ becomes $\frac{d^N f}{dA^{\dagger\ N}=a\frac{d^{N-1}f}{dA^{\dagger\ N}}}$. If one choses $f(0) = 1$ then $\frac{d^N f}{dA^{\dagger\ N}}(0) = a^N$ and $f(A^\dagger) = e^{aA^\dagger}$

If we now choose $A = \tilde{A} + a$, then $\tilde{A}$ also obeys $[\tilde{A}, \tilde{A}^\dagger] = 1$ and the vacuum state for $\tilde{A}$ will obey $A|\tilde{0}\rangle = a|\tilde{0}\rangle$.

We will choose the state for the field $\tilde{\phi}$ such that $A_{\omega+\nu}|\psi >= \alpha\delta(\nu)$. We will call $A_\nu = A_{\omega+\nu} - \alpha\delta(\nu)$.

2) Mirror pressure;

The interction between the field and the mirror, which is what you are measuring, is, on the one had, the reflection by the mirror of the light, and on the other hand the field influences the mirror by by the radiation pressure of the radiation on the mirror.

If one has a Lagrangian for the field, say

$$\mathcal{L} = 1/2\int(\partial_t\phi^2) - (\partial_x\phi)^2 dx, thenthiscanbe \tag{4}$$

written in terms of the metric (the distance function in spacetime) g^{ij} as

$$\mathcal{L} = \frac{1}{2}\int\frac{1}{\sqrt{-g^{tt}g^{xx} + (g^{tx})^2}}\int\left[(g^{tt}\partial_t\phi)^2 + g^{xx}(\partial_x\phi)^2 + 2g^{tx}\partial_t\phi\partial_x\phi\right]dx \tag{5}$$

Then the components of the energy momentum tensor are

$$E = \delta_{g^{tt}}\mathcal{L} \tag{6}$$

$$P = \delta_{g^{xx}}\mathcal{L} \tag{7}$$

$$J = \frac{1}{2}\delta_{g^{tx}}\mathcal{L} \tag{8}$$

$$\tag{9}$$

evaluated with $g^{11} = 1$, $g^{xx} = -1$ and $g^{tx} = 0$. If one does this, then one gets the conservation laws

$$\partial_t E - \partial_x J = 0 \tag{10}$$

$$\partial_t J + \partial_x P = 0 \tag{11}$$

These are the energy/momentum conservation laws, with the momentum being the same as the Energy flux J. P is the pressure, E is the energy density, and J is the energy flux or the momentum density. The pressure become

$$P = \pm\frac{1}{4}(\partial_t\phi^2 + (\partial_x\phi) \tag{12}$$

If P is positive, the pressure exerts a force outwards on any boundary.

3) Particle flux.

A particle detector acts by the field exiciting some detector from its ground state to an excited state with Energy ϵ. The excitation operator will be some creation operator time $e^{i\epsilon t}$. The interaction between the detector and the field is given by a product between the creation operator of the detector times the field, which one has to integrate over time. This picks out (for a static detector) the components of the field which go as $e^{-i\lambda t}$ at the position of the detector. The particle density is essentially the norm

$$\rho_N = \frac{i}{2}(\Phi_+^\dagger \partial_t Phi_+ - \partial_t \Phi_+^\dagger \Phi_+). \tag{13}$$

where Φ_+ are the componets of Φ which go as $e^{-i\lambda t}$ with $\lambda > 0$ (they are the parts of the field which could excite the detector). since particles should be conserved the particle flux is

$$j_N = -\frac{i}{2}((\Phi_+^\dagger \partial_x Phi_+ - \partial_x \Phi_+^\dagger \Phi_+) \tag{14}$$

and $\partial_t \rho_N + \partial_x j_n = 0$.

4)In this paper, I assume that the boundary conditions on the mirror are the Neumann boundary condition for the field. These are for example the conditions obeyed by the magnetic field at the surface of a perfect conductor. This condition is that the normal spatial derivative of the field at the surface is equal to zero. One could also use Dirchlet conditions, which is where the value of the field on the surface is zero. Both give the same answers, although the outgoing field has opposite signs in the two cases. This does not change the final result. (One could even require Robin conditions in which the a fixed combination of value plus derivative be zero on the boundary. Again, it makes no change to the result, although the conditions are messier, and leads to a phase shift on the boundary for a plane-type wave).

5) Beam splitter

I assume that the Beamsplitter is the simplest kind where the field coming in to the beam splitter is from one direction splits the beam in two, with a relative

+ sign between them, and with a - sign between the two when the the beam comes in in the other direction. I also assume tha the system is symmetric– if one reflects the out beams back with the reflecting Neumann mirros being equidistant from the splitter, the results will come back out the ssame ports they want in, with the same sign. The beamsplitter could insert a variety of phases on the two possibilities (eg, such that the splitter inintroduced phases of various sorts into the varous beams (with the condition that the total norms are preserved). Again, this could well alter the details (eg the "dark" output could be the bright input– eg, with perfect equispaced mirrors, if one shone a light into one port, it could all come back out of the other port.) The bright input port would be the dark output port. Dark input implies dark output with perfect symmetry for the mirrors is the easiest for me to analyse.

Laser Interferometer measurement of force on a mirror

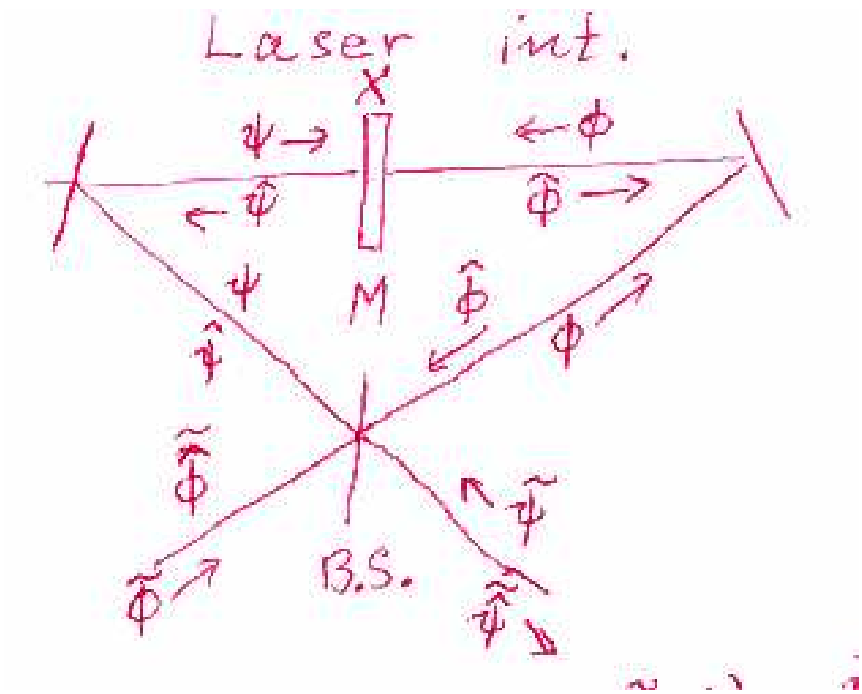

Figure 1: laser interferometer measurement on mirror M with beamsplitter BS

$$\phi(t-x) = \frac{1}{\sqrt{2}}(\tilde{\phi}(t-x) + \tilde{\psi}(t-x)) \tag{15}$$

$$\psi(t-x) = \frac{1}{\sqrt{2}}(\tilde{\phi}(t-y) - \tilde{\psi}(t-y)) \tag{16}$$

Neumann boundary conditions at $x = X(t);\ y = -X(t)$

$$\begin{aligned}
\partial_x(\hat{\phi}(t+x) + \phi(t-x))|_{x=X(t)} &= 0 \qquad (17)\\
\partial_y(\hat{\psi}(t+y) + \psi(t-y))|_{y=-X(t)} &= 0 \qquad (18)\\
\hat{\phi}(t) &= \phi(t-2X(t)) \qquad (19)\\
\hat{\psi}(t) &= \phi(t+2X(t)) \qquad (20)
\end{aligned}$$

(we assume that $X(t)$ is very small and $X(t)$ is lowly varying so that $X(t + 2X(t)) \approx X(t)$)

$$\tilde{\hat{\phi}}(t+x) = \frac{1}{\sqrt{2}}(\hat{\phi}(t+x) + \hat{\psi}(t+x)) \qquad (21)$$

$$\tilde{\hat{\psi}}(t+y) = \frac{1}{\sqrt{2}}(\hat{\psi}(t+y) - \hat{\phi}(t+y)) \qquad (22)$$

$$(23)$$

Then, after reflection at the mirror,

$$\begin{aligned}
\tilde{\hat{\psi}}(t+y) &= \frac{1}{2}\Big((\tilde{\psi}(t+y-2X(t)) + \tilde{\phi}(t+y-2X)) \qquad (24)\\
&+ (\tilde{\psi}(t+y+2X(t)) - \phi(t+y+2X(t))\Big) \qquad (25)\\
&\approx \frac{1}{2}(2\tilde{\psi}(t+y) - 2\partial_t\tilde{\phi}(t+y)(X_0 + 2\hat{X}(t)) \qquad (26)
\end{aligned}$$

Where we have retained only terms to lowest order in $X(t)$. We assume that $X(t) = x_0 + \hat{X}(t)$ where $< \hat{X}(t)^2 ><< x_0^2 << |\alpha|^2$ and x_0 is a c-number.

We now assume that

$$\begin{aligned}
\tilde{\phi}(t-x) &= \frac{e^{-i\omega(t-x)}}{\sqrt{2\pi\omega}}(\alpha + A(t-x))e^{-i\nu(t-x)}d\nu) + \mathrm{HC} \qquad (27)\\
\tilde{\psi}(t-y) &= \frac{e^{-i\omega(t-y)}}{\sqrt{2\pi\omega}}B(t) + \mathrm{HC} \qquad (28)
\end{aligned}$$

where $|\alpha| >> 1$ and where we have used $\omega + \nu \approx \omega$. ν will be the fourier frequencies of the the mirror motion , which will be assumed to be small (less then 1KHz) while ω is an optical laser frequency (10^15 Hz). We will keep only terms linear in A, B, X with no cross terms between them.

α is the coherent C-number amplitude of the laser beam and will be assumed to be real, A_ν and B_ν are the annihilation operators around that coherent amplitude for the $\tilde{\phi}$ and $\tilde{\psi}$ fields. (In more conventional notation $A_\nu = A_{\omega+\nu}$ and similarly for B_ν, the annihilation operators at the frequency around the C-number coherent laser output. Then the output from the ψ port, the $\tilde{\hat{\psi}}$ field, will be

$$\tilde{\hat{\psi}}(t+y) \approx \frac{e^{-i\omega(t+y)}}{\sqrt{2\pi\omega}}\Big(-i\omega(x_0 + \hat{X}(t))\alpha + B(t)\Big) + \mathrm{HC} \qquad (29)$$

A particle detector is a device that gets excited when it interacts with a quantum field in an excited state. If the detetor has energy E the probability is propoportional to the operator

$$\int \phi(t',x)e^{iEt'}dt \tag{30}$$

This selects out the terms in $\phi(t,x)$ which go as e^{-iEt} and it is htus the $\hat{\psi}_+(t+x)$ terms which could excite a detector (assuming that the detector is at rest and not accelerated.)

The particle number flux is given by

$$\begin{aligned} NF &= \frac{i}{2}(\tilde{\hat{\psi}}^\dagger_+(t+y)\partial_y\tilde{\hat{\psi}}_+(t+y) - \partial_y\tilde{\hat{\psi}}^\dagger{}_+(t+y)\tilde{\hat{\psi}}_+(t+y) \quad (31)\\ &\approx \omega(\tilde{\hat{\psi}}^\dagger{}_+(t+y)\tilde{\hat{\psi}}_+(t+y) \quad (32)\end{aligned}$$

where $\tilde{\hat{\psi}}_+$ is the part of $\tilde{\hat{\psi}}_+$ which goes as $e^{-i\omega(t+y)}$, or

$$\tilde{\hat{\psi}}_+ = \frac{e^{-i\omega(t+y)}}{\sqrt{2\pi\omega}}\left(-i\omega(x_0+\hat{X}(t))\alpha + B(t)\right) \tag{33}$$

In the number flux, we keep terms to only first order in $\hat{X}$ and B to give

$$NF \approx \frac{1}{2\pi}\left(\alpha^2\omega^2(x_0^2+2x_0\hat{X}) - \alpha x_0 i(B(t+y) - B^\dagger(t+y))\right) \tag{34}$$

We now need to look at the behaviour of $\hat{X}(t)$ and its fourier transform. Its equation of motion will be

$$M\partial_t^2\hat{X} = P\mathcal{A} + F(t) \tag{35}$$

where P is the pressure on the mirror due to the ϕ and ψ fields, A is the effective area of the mirror that the pressure is acting on, and $F(t)$ is a classical force that acts on the mirror and is what we want to measure.

The energy density and pressure in the rest frame are given by

$$\begin{aligned} P &= \frac{1}{4}\Big((\partial_t\phi_T(t,x))^2 + (\partial_x\hat{\phi}_T(t,x))^2) \quad (36)\\ &- (\partial_t\phi_T(t,y))^2 - (\partial_y\hat{\psi}_T(t,y))^2)\Big)\,|_{x=-y=X(t)} \quad (37)\\ E &= \frac{1}{4}\left((\partial_t\phi_T(t,x))^2 + (\partial_x\phi_T(t,x))^2\right) \quad (38)\\ &+ (\partial_t\phi_T(t,y))^2 + (\partial_y\psi_T(t,y))^2))\,|_{x=-y=X(t)} \quad (39)\end{aligned}$$

Here

$$\begin{aligned} \phi_T(t,x) = \phi(t-x) + \hat{\phi}(t+x) = \phi(t-x) + \phi(t+x-2X(t+x)) \quad (40)\\ \psi_T(t,x) = \psi(t-y) + \hat{\psi}(t+y) = \psi(t-y) + \psi(t+y+2X(t+y)) \quad (41)\end{aligned}$$

Now, at $x = X;\ y = -X$, the Neuman boundary condition is that the derivative with respect to x and y are zero, so only the time derivative terms are non-zero

$$\begin{aligned} P &\approx \frac{1}{4}((\partial_t\phi(t - X(t)) + \partial_t\phi(t - X(t))(1 - 2\partial_t X(t)))^2 && (42)\\ &- (\partial_t\psi(t + X(t)) + \partial_t\psi(t + X(t))(1 + 2\partial_t X(t)))^2) && (43)\\ &= \frac{1}{4}((\partial_t\phi(t))^2 - (\partial_t\psi)^2) - 4(\partial_t\phi(t)\partial_t^2\phi(t) + \partial_t\psi(t)\partial^2\psi(t))X && (44)\\ &- 4(\partial_t\phi(t)^2 + (\partial_t\psi(t))^2)\partial_t\hat{X}(t) && (45) \end{aligned}$$

Given the expression for $\phi(t)$ and $\psi(t)$ and keeping only the lowest order in A, B, X terms, and nelecting terms which have a time dependence of around $e^{\pm i2\omega(t)}$ since such a rapidly oscillating force at frequencies of 10^15 would not move the mirror at all, we get

$$P = \frac{1}{4}\left[2\left(2\frac{\omega}{2\pi}\alpha(B(t) + B^\dagger(t))\right) - \frac{\omega}{2\pi}|\alpha|^2 4(\partial_t\hat{X})\right] \quad (46)$$

Note that both the "classical" radiation pressure due to the coherent state, and that due to the quantum field coming in from the laser (The $A, A^\dagger$ terms) cancel out, leaving the quantum field coming in from the ψ port (the $B, B^\dagger$ terms). and the reaction force of the field back onto the oscillator, the $\partial_t\hat{X}$ term is a damping force on the mirror.

Note that $\partial_t\phi(t)\partial_t^2\phi(t) = \frac{1}{2}\partial_t(\partial_t\phi(t))^2$ which only has terms which go as $e^{\pm i2\omega t}$ and will have no effect on the mirror. The terms in the pressure which go as $B, B^\dagger$ are quantum radiation pressure terms, or fluctuating pressure forces usually ascribed to the shot noise in the number photons hitting the mirror. The shot noise due to quantum fluctuations in the laser cancel out, and it is only the quantum noise coming from the dark port (the $\tilde{\phi}$ port) that contribute to the radiation pressure noise, and this is proportional to $B(t) + B\dagger$.

The equation of motion of the mirror is then

$$\begin{aligned} M\partial_t^2\hat{X} &\approx P\mathcal{A} + F(t) && (47)\\ &= \mathcal{A}\frac{\omega}{2\pi}\alpha((B(t) + B^\dagger(t))) - \frac{\omega}{2\pi}\alpha^2\partial_t\hat{X}(t) + F(t) && (48)\\ \hat{X}_\nu &= \frac{1}{-M\nu^2 - i\nu\gamma}\left(\mathcal{A}\frac{\omega}{2\pi}\left(\alpha(\int(B_\nu + B^\dagger_{-\nu})d\nu + F_\nu\right)\right) && (49)\\ \gamma &= \mathcal{A}\frac{\omega}{2\pi}\alpha^2 && (50) \end{aligned}$$

For reasonable powers of the laser, γ will be small (less than values of $M\nu$ for values of ν which will be measureable.) Thus $\hat{X}_\nu \approx -\frac{1}{M\nu^2}(F_\nu + \mathcal{A}\frac{\omega}{2\pi}\alpha(B_\nu + B^\dagger_\nu)$

Inserting this into the output particle flux we get

$$\begin{aligned} NF &\approx \frac{1}{2\pi}\left(\alpha^2\omega^2(x_0^2 + 2x_0\hat{X}) - 2\alpha x_0 i(B(t + y) - B^\dagger(t + y))\right) && (51)\\ &= \frac{1}{2\pi}\left(\alpha^2\omega^2(x_0^2 + 2x_0\hat{F}(t)\right) && (52) \end{aligned}$$

$$+ \quad \left[-2\alpha^2 x_0 \mathcal{A} \int \omega \frac{(B_\nu + B^\dagger_{-\nu})}{-M\nu^2} e^{-i\nu(t+y)} d\nu + \alpha x_0 i (B_\nu - B^\dagger) e^{-i\nu(t+y)} \right] \tag{53}$$

The first term in the large parentheses is a classical expression and is the only terms remaining if the inputs into the ψ port have zero expectation value for B (for example is the vacuum state). We thus have

$$NF- < NF >= \left[\frac{-2\alpha^2 x_0^2 \mathcal{A} \int \omega (B_\nu + B^\dagger_{-\nu})}{-M\nu^2 e^{-i\nu(t+y)} d\nu + \alpha x_0 i (B_\nu - B^\dagger) e^{-i\nu(t+y)} d\nu} \right] \tag{54}$$

and the quantum noise is

$$< (NF- < NF >)^2 >=< \left[-2\alpha^2 x_0^2 \mathcal{A} \int \omega \frac{(B_\nu + B^\dagger_{-\nu})}{-M\nu^2} e^{-i\nu(t+y)} d\nu + \alpha x_0 i (B_\nu - B^\dagger) e^{-i\nu(t+y)} d\nu \right]^2 > \tag{55}$$

Defining

$$S_\nu = -2\alpha^2 x_0 \mathcal{A} \frac{\omega}{-M\nu^2} + i\alpha x_0 \tag{56}$$

we get

$$< (NF- < NF >)^2 >= S_\nu B_\nu + S^*_\nu B^\dagger_{-\nu} = |S_\nu| (e^{i\delta_\nu} B_\nu + e^{-i\delta_\nu} B^\dagger_{-\nu}) \tag{57}$$

If the state is a vacuum state, then

$$\langle 0|_B < (NF- < NF >)^2 |0\rangle_B = \int |S_\nu|^2 d\nu \tag{58}$$

However, if we define a two mode squeezed state by

$$B_\nu = cosh(r) \tilde{e}^{i\delta_\nu} B_\nu + \sinh(r) e^{-i\delta_\nu} \tilde{B}^\dagger_{-\nu} \tag{59}$$

$$B_{-\nu} = cosh(r) e^{i\delta_\nu} \tilde{B}_{-\nu} - sinh(r) e^{-i\delta_\nu} \tilde{B}^\dagger_\nu \tag{60}$$

and define the squeezed vaccum by $\tilde{B}_\nu |0\rangle_r = \tilde{B}_{-\nu} |0\rangle_r = 0$ then

$$\langle 0|_r (B_\nu e^{i\delta_\nu} + B^\dagger_{-\nu})^2 |0\rangle_r = e^{-2r} \tag{61}$$

and one can decrease the quantum noise by an arbitrary factor for all frequencies.

Ie, by an appropriate squeezing of the state entering into the input port of the interferometer, one can reduce the quantum noise a much as one desires. Since each B_ν is independent of the others, this can be done independently (in principle) at each value of ν giving a quantum noise as small as desired.

Note that δ_ν is a function of ν (the real part of S_ν dies as $\frac{1}{\nu^2}$, with respect to the imaginary part or $\tan(\delta_\nu)$ grows as ν^2 The real part comes from the radiation pressure noise, while the imaginar7y part comes from the "shot" noise at the readout.

Of course if $< \hat{X}^2 >$ becomes of order x_0^2, the linearization approximations we made fail, and one would have to do more work.