A Simple Landau-Zenner type model

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A simple example of the Landau-Zenner type transition is presented which is solveable in terms of elementary functions.

In 1932 Zenner[2], improving on a simultaneous perturbative treatment by Landau[1], was presented for the transition between weakly interacting levels of a two level system, to examine the transitions between the energy levels of this system for a time dependent Hamiltonian. The solution, in terms of Weber functions, is not very transparent. Let us present a model which is solvable in terms of elementary functions instead, which is at least of some pedagogic value.

The Hamiltonian for the system, being a two level system can be written in terms of the Pauli matrices, and we shall write it as

$$H(\theta) = f(\theta)\sigma_z\Theta(-\theta) + (cos(\theta)\sigma_x + sin(\theta)\sigma_x)\Theta(\theta)\Theta(\pi-\theta) - f(\pi-\theta)\Theta(\theta-\pi) \tag{1}$$

where $f(\theta)$ is taken to be some decreasing function of its (negative) argument such that $f(0) = 1$ and $\partial_\theta f(0) = 0$. A simple one would be to have $f(\theta) = -\theta$ for $\theta << 0$. (It will actually not matter what the form is of $f(\theta)$ but this corresponds more closely with the usual form of the problem). The Energy Eigentates are then given with

$$E(\theta) = \pm(f(\theta)\Theta(-\theta) - f(\pi-\theta)\Theta(\theta-\pi) + \Theta(-\theta(\pi-\theta)) \tag{2}$$

For example, if we choose $f(\theta) = 1 - \frac{\theta^2}{1-\theta}$ the energy eigenstates would look as in figure 1.

If we choose our states such that

$$\sigma_z|0\rangle = -|0\rangle; \qquad \sigma_z|1\rangle = |1\rangle \tag{3}$$

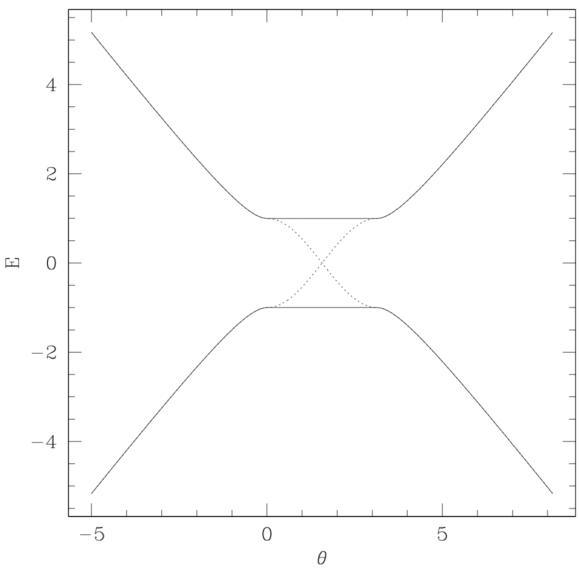

FIG. 1: The energies eigenvalues of the Hamiltonian. The dotted line is the energies if the σ_x term in the Hamiltonian is set to 0, removing the coupling between the states.

Then the eigenstates are

$$|0_\theta\rangle = cos(\theta/2)|0\rangle + \sin(\theta/2)|1\rangle \tag{4}$$
$$|1_\theta\rangle = cos(\theta/2)|1\rangle - \sin(\theta/2)|0\rangle \tag{5}$$

for $0 < \theta < \pi$.

Now assume that $\theta(t) = t/T$ for some T. A large value of T would corresponnd to an adiabatic transition (slow transition) while $T << 1$ would correspond to a diabatic transition (fast transition).

Define the state of the system to be given by

$$|\phi\rangle = \alpha(t)|0\rangle + \beta(t)|1\rangle \tag{6}$$

which will have the Schroedinger equation

$$t < 0 \text{ or } t > \pi \tag{7}$$
$$i\dot{\alpha} \;=\; (-f(\theta(t))\Theta(-t) + f(\theta(\pi - t))\Theta(t - \pi))\alpha \tag{8}$$
$$i\dot{\beta} \;=\; -(-f(\theta(t))\Theta(-t) + f(\theta(\pi - t))\Theta(t - \pi)\beta \tag{9}$$
$$t > 0 \text{ and } t < \pi \tag{10}$$
$$i\dot{\alpha} \;=\; -cos(\theta(t))\alpha + sin(\theta(t))\beta \tag{11}$$
$$i\dot{\beta} \;=\; sin(\theta(t)\alpha cos(\theta)\alpha \tag{12}$$

where $\dot{} = \partial_t$. Except between $0 < t < \pi$ these equations are trivial to solve ($|\alpha|$ and $|\beta|$ are constant), so we will concentrate on on the region between 0 and π.

Rewriting these equations in terms of $\alpha + i\beta$ and $\alpha - i\beta$ (where, since α and β will be complex there are not complex conjugate of each other), we get

$$i\partial_t(\alpha + i\beta) = -e^{i\theta(t)}(\alpha - i\beta) \tag{13}$$
$$i\partial_t(\alpha - i\beta) = -e^{-i\theta(t)}(\alpha + i\beta) \tag{14}$$

Defining

$$Z_1 = e^{-i\theta(t)/2}(\alpha + i\beta) \tag{15}$$
$$Z_2 = e^{i\theta(t)/2}(\alpha - i\beta(t) \tag{16}$$

to give

$$i\partial_t Z1 - \dot{\theta}(t)/2Z_1 = -Z_2 \tag{17}$$
$$i\partial_t Z_2 + \dot{\theta}(t)/2Z_2 = -Z_1 \tag{18}$$

from which we find, since $\dot{\theta}(t)$ is constant, that both Z_1 and Z_2 obey the same equation

$$-\partial_t^2 Z - \dot{\theta}(t)^2 Z = Z \tag{19}$$

or

$$Z = ae^{i\omega t} + be^{-i\omega t}. \tag{20}$$

However, the boundary condition at $t = 0$ will differ. Let us assume that we started with the system in the ground state of the Hamiltonian for $t < 0$, or $\alpha(0) = 1 \quad \beta(0) = 0$. The Schroedinger equation will have $|\alpha| = 1$ and $|\beta| = 0$ for all $t < 0 >$. However

$$\partial_t Z_1(0) = -1Z_2(0) + \dot{\theta}(0)/2Z_1 = -1 + \frac{1}{2T} \tag{21}$$
$$\partial_t Z_2(0) = -1Z_1 - \dot{\theta}(0)/2Z_1 = -1 - \frac{1}{2T} \tag{22}$$

from which

$$a_1 + b_1 = 1; \qquad a_2 + b_2 = 1 \tag{23}$$

$$i\omega(a_1 - b_1) = -1 + \frac{1}{2T}; \qquad i\omega(a_2 - b_2) = -1 - \frac{1}{2T} \tag{24}$$

$$a_1 = \frac{2iT\omega - 2T + 1}{i2\omega} \qquad a_2 = \frac{2iT\omega 1 - 2T - 1}{i2\omega} \tag{25}$$

$$b_1 = -\frac{2iT\omega + 2T - 1}{2\omega} \qquad b_2 = -\frac{2iT\omega + 2T + 1}{2\omega} \tag{26}$$

$$\tag{27}$$

Going to $t = \pi T$ we have

$$\alpha(\pi t) = (iZ_1 - iZ2)/2 = \frac{-sin(\pi\sqrt{4T^2+1}/2)}{2\sqrt{4T^2+1}} \tag{28}$$

$$\beta(\pi t) = (Z_1 + Z_2)/2 = \frac{(\sqrt{4T^2+1} - 2T)e^{-i\pi\sqrt{4T^2+1}/2} + (\sqrt{4T^2+1} + 2T)e^{i\pi\sqrt{4T^2+1}/2}}{2\sqrt{4T^2+1}} \tag{29}$$

as the generic solution for arbitrary speed of transition.

In the limit at $T \to 0$, we have

$$\alpha(\pi) \approx -1 + 2T^2 + O(T^4) \tag{30}$$

$$\beta(\pi) \approx 2iT + O(T^2) \tag{31}$$

Ie, in the sudden transition (T¡¡1), system will remain in the state $|0\rangle$ which is the higher energy eigenstate of the Hamiltonian after the transition. Ie, this would be as if the σ_x term in the Hamiltonian were 0, and the σ_z had a continuous term σ_z connecting $f(\theta)$ to $-f(\pi - \theta)$. if T is small, the probability that there was a transition to the other , the ground state, is proportional to T^2.

In the case that $\tau = \frac{1}{T}$ goes to zero, the adiabatic limit, we find that

$$\alpha \approx \frac{sin(\pi T)}{4T} + O(1/T^2) \tag{32}$$

$$\beta \approx \frac{(\frac{1}{4T})e^{-i\pi\sqrt{4T^2+1}/2} + (4T + \frac{1}{4T})e^{i\pi T}(1 + i\frac{\pi}{4T})}{4T} = e^{i\pi T}(1 + i\frac{\pi}{8T} - \frac{(1+\frac{\pi^2}{8})}{16T^2} + e^{-i\pi/T}\frac{1}{16T^2} + O(1/T^3) \tag{33}$$

Ie, after the transition, the probability is large that system is in the $|1\rangle$ state, which is the lower energy eigenstate after the transition. Unlike the Landau-Zenner case however, the probability of being in the higher level falls as $1/T^2$ rather than exponentially in T.

In figure 2 is plotted the probability of the system remaining the state $|0\rangle$ as a functionof the transition time. Figure2 gives the complementary probability of making the transition to the state $|1\rangle$ which is the ground state of the Hamiltonian after the transition.

The above assumes that the energy at the transition was 1. One can scale this solution, by taking $t \to \epsilon t$ and $T \to \epsilon T$. where epsilon is the half energy difference between the upper and lower states during the transition.

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[1] L. Landau . "Zur Theorie der Energieubertragung. II". Physikalische Zeitschrift der Sowjetunion. 2: 46–51.(1932)
[2] C. Zenner "Non-Adiabatic Crossing of Energy Levels". Proceedings of the Royal Society of London A. 137 (6): 696-702 (1932)

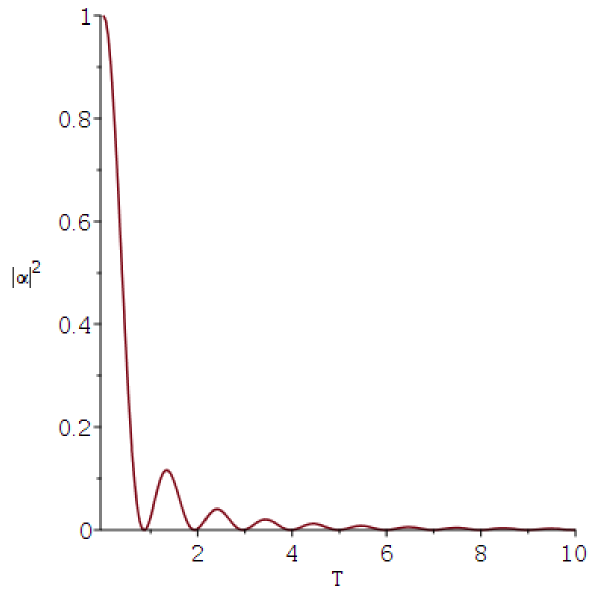

FIG. 2: The probability of remaining in the $|0\rangle$ state

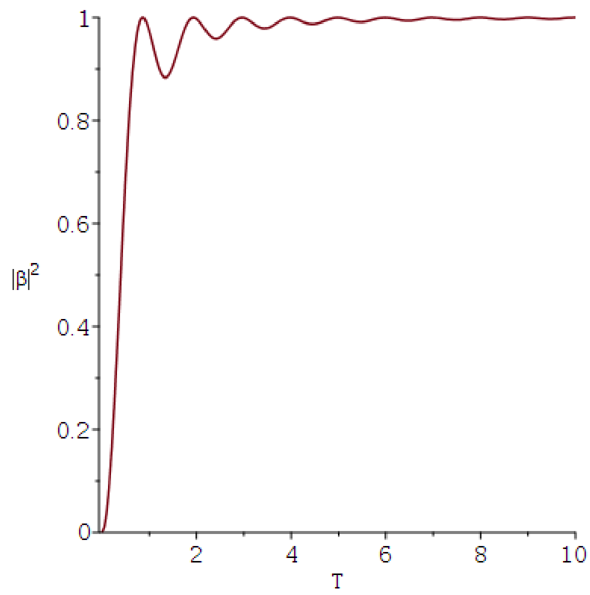

FIG. 3: The probability of making a transition to the orthogonal state during the regime of level interaction. The transition takes place around T=.4 (50-50 chance of being in the lower or higher energy eigenstate)