Physics 501-20

Interaction representation and Detectors

Let us say that we have a system with a Hamiltonian which we can write in two parts

$$H = H_0 + H_i \tag{1}$$

where H_0 will be a Hamiltonian for which it is easy to solve the Heisenberg equations of motion for the operators.

In the Schroedinger representation, the equation for the state $|\psi, t\rangle$ obeys

$$|\psi, t\rangle = U\,|\psi, 0\rangle \tag{2}$$

where $|\psi, 0\rangle$ is the intial Schroedinger state. while U is the opertor which solves the equation

$$i\partial^t U = HU \tag{3}$$

This is formally solved by

$$U = \mathbf{T}exp(-i\int_0^t Hdt'). \tag{4}$$

where $\mathbf{T}$ is is the "time ordering operator" so that

$$\mathbf{T}exp(-i\int_0^t H(t')dt') = \lim_{\Delta t\to 0} \Pi^0_{n=t/Deltat-1} e^{iH(t_n)\Delta t} \tag{5}$$

where $t_n = (n + \frac{1}{2})\Delta t$. and H is assumed to have have an explicit time dependence.

Now consider the operator

$$U_0 = \mathbf{T}e^{-i\int_0^t H_0 dt'} \tag{6}$$

where H_0 is assumed to be independent of time. It obeys

$$\partial_t U_0 = -iH_0U_0 \tag{7}$$

Consider the time dependent state

$$|\psi_I, t\rangle = U_0^\dagger U\,|\psi, 0\rangle \tag{8}$$

Then

$$i\partial_t\,|\psi_I, t\rangle = U_0^\dagger(-H_0 + H)U\,|\psi, 0\rangle \tag{9}$$

$$\tag{10}$$

Since $H = H_0 + H_I$ where H_I is the interaction Hamiltonian, we have

$$i\partial_t\,|\psi_I, t\rangle = U_0^\dagger(H_I)U_0\,|\psi_I, t\rangle \tag{11}$$

Now if H_I is a sum of polynomials of whatever fundamental operators there are for the system, (eg, Ψ, Π, σ_w,...), then

$$U_0^\dagger H(\Psi,\Pi,\sigma_w...)U_0 = H(U_0^\dagger \Psi U_0, U_0^\dagger \Pi U_0, U_0\sigma_w U_0, ...) \tag{12}$$

Ie, H_i will be a function of the explicitly time dependent dynamic operators.

Now, assuming H_0 is not explicitly time dependent, and neither is the operator A,

$$i\partial_t(U_0^\dagger A U_0)) = U_0^\dagger(-H_0 A + A H_0)U_0 = [U_0^\dagger A U_0, H] \tag{13}$$

But this is just the operator equation for the Heisenberg representation of the operator $A_H = U_0^\dagger A U_0$ if the Hamiltonian is H_0. Thus the Interaction representation obeys a Schroedinger equation with the interaction Hamiltonian where the operators in that Hamiltonian are replaced by the explicitly time dependent dynamical operators which obey the Heisenberg equations of motion with H_0.

One thus chooses H_0 so that the Heisenberg equations of motion are easily solvable. For example, we choose it to be the free equations of motion for the Hamiltonian, or the single particle dynamics of the two level system.

Let us take as an example a quantum field theory for a massive three spatial dimensioned field ϕ. The free Hamiltonian is then

$$H_{0\phi} = \frac{1}{2}\int (\Pi(x)^2_\nabla \Phi(x)\cdot\nabla\Phi(x) + m^2\Phi(x)^2)dx \tag{14}$$

We know that we can choose a set of modes $\phi_i(t,x)$ which obey the field equation

$$\partial_t^2\phi_i(t,x) - \nabla^2\phi_i(t,x) + m^2\phi_i(t,x) = 0 \tag{15}$$

$$\pi_i(t,x) = \partial_t\phi_i(t,x) \tag{16}$$

such that inner product

$$<\phi_i,\phi_j>= i\int(\phi_j(t,x)^*\pi_i(t,x) - \pi_j(t,x)^*\phi_i(t,x))d^Dx = \delta_{ij} <\phi_j^*, phi_i>= 0 \tag{17}$$

(if they are normalisable modes, or if the subscript is continuous, then it would be the Dirac delta ($\delta(i-j)$). Then

$$\Phi(t,x) = \sum_i (A_i\phi_i(t,x) + A_i^\dagger\phi_i^*(t,x)) \tag{18}$$

$$\Pi(t,x) = \sum_i (A_i\partial_t\phi_i(t,x) + A_i^\dagger\partial_t\phi_i^*(t,x)) \tag{19}$$

If we want to choose Hamiltonian diagonalisation as our definition of the modes, so that the vacum state is the lowest energy state, then $\phi_i(t,x)$ should all be chosen so that $\phi_i(t,x)$ is a sum of only functions whisch are made up of temporal fourier terms which go as $e^{-i\omega t}$. One set of modes are

$$\phi_k(t,x) = \frac{e^{-i(\omega t - k\cdot x)}}{\sqrt{2\omega(2\pi)^D}} \tag{20}$$

with $\omega = +\sqrt{k^2+m^2}$.

Similarly, if one of the dynamical systems is a two level system, with operators $\sigma_x, \sigma_y, \sigma_z$, with Hamiltonian

$$H_{0\sigma} = \frac{1}{2}E\sigma_z \tag{21}$$

then the Heisenberg solutions are

$$\sigma_z(t) = \sigma_{z0} \tag{22}$$
$$\sigma_-(t) = \sigma_{0-}e^{-iEt} \tag{23}$$
$$\sigma_-^\dagger = \sigma_{0-}^\dagger e^{iEt} \tag{24}$$
$$\sigma_x(t) = \sigma_-(t) + \sigma_-^\dagger(t) \tag{25}$$
$$\sigma_y(t) = -i(\sigma_-(t) - \sigma_-^\dagger(t)) \tag{26}$$

Let us say that we have an interaction Hamiltonian which looks like

$$H_I = \epsilon\sigma_x\Pi(x_0) \tag{27}$$

Then in the interaction picture, the equation of the state is

$$i\partial_t |\psi,t\rangle = \epsilon(\sigma_{0-}e^{-iEt} + \sigma_{0-}^\dagger e^{iEt})(\sum_i A_i\partial_t\phi_t(t,x) + A_i^\dagger\partial_t\phi_t(t)^*)) |\psi,t\rangle \tag{28}$$

Writing $|\psi,t\rangle = |\psi,0\rangle + \epsilon\,|\delta\psi,t\rangle + \epsilon^2....$ and keeping only terms to first order in ϵ, we have

$$\epsilon\,|\delta\psi,t\rangle = \epsilon\int_0^t (\sigma_{0-}e^{-iEt'} + \sigma_{0-}^\dagger e^{iEt'})(\sum_i A_i\partial_t'\phi_i(t',x) + A_i^\dagger\partial_t\phi_i(t')^*)) |\psi,0\rangle)dt' \tag{29}$$

Now let us assume that the state $|\psi,0\rangle = |\phi\rangle\,|{\downarrow}\rangle$ where $|{\downarrow}\rangle$ is the -1 eigenvalued state of σ_z.Then $\sigma_-\,|{\downarrow}\rangle = 0$. Also, as we allow $t\to\infty$, only the $\int\phi_i(t,x0)e^{iEt}dt$ will survive, because only $\phi_i(t,x_0)$ has non-zero fourier transform with positive E. Thus for large t, we have

$$\epsilon\,|\delta\psi,t\rangle = \epsilon\,|{\uparrow}\rangle\sum_i\int\phi_i(t',x_0)e^{iEt'}dt'A_i\,|\phi,0\rangle \tag{30}$$

If $|\phi,0\rangle = |0\rangle$, then the A_i destroy this state, and the $|\delta\psi,t\rangle = 0$. There is no excitation of the detector.

If $|\phi,0\rangle = A_j^\dagger\,|0\rangle$, then the detector has a non zero amplitude of being excited.

$$\epsilon\,|\langle\rangle\,\delta\psi,t) = \epsilon\int_0^t\phi_i(t',x_0)e^{iEt'}dt'\,|0\rangle\,|{\uparrow}\rangle \tag{31}$$

and the probability of finding the detector excited will be

$$Prob_\uparrow = \epsilon^2|\int_0^t\phi_i(t',x_0)e^{iEt'}dt'|^2 \tag{32}$$

Ie, only if the mode has a non-zero amplitude of being at x_0 at some time can there be any probability of the detector being excited. And only if the mode has a component of its time dependence with frequency E will this probability be non-zero.

We note that this is just energy conservation– the mode has to have a component of energy E if it is to excite the detector whose excited state has energy E above its lowest state.