Quantum Ligo.

Laser Interferometer Gravity wave observatory.

a) What is gravitational wave

(the term gravity wave refers to waves on the surface of a fluid (like ocean waves) whose restoring force is gravity (rather than, say, surface tension)

In General Relativity all effects of gravity are the result of changes in either time or distances. Thus the gravity on earth is the result of time flowing more slowly near a massive body than further away.

Gravitational waves are the result of distances changing in time. - Distances changing not because of motion of the two objects (eg by $F = ma$) but the actual distance changing on its own.

It is a transverse wave (the changes in distance are perp. to the direction of travel, which travels at the velocity of light.

There are two forms of gravitational wave detectors.

In the original idea (by Weber) solid objects were used. The atoms in a solid object like to be a certain distance apart (electron orbitals) and when that distance changes, the atoms push or pull against that change. Those pushes or pulls start them moving & that motion remains after the wave has gone by. Because of the form of the grav. wave, [The changes in distance are opposite in perp. directions – ie they do not change the volume of space – it is the shear modulus (the reaction of the solid body to changes in shape) that is important. It starts the body vibrating & that vibration remains after the grav wave has gone by. The experimentalist then detects that vibration.

Such detectors are not very sensitive. The velocity of shear wave sound would need to be near the velocity of light for these detectors ~~to~~ to be efficient.

The second - originally proposed & studied by R Forward and R. Weiss, is to use an interferometer to measure the ~~dist~~ difference in distance between a central "free" mass and ~~dis~~ two distant masses.

The strength of a gravity wave is reflected in how much of a fractional change of distance it causes.

Clearly the longer the interferometer the larger the actual change in distace between the masses, and the larger the signal.

Gravity wave detectors are sensitive ~~to~~ at around 100 Hz. at which the wavelength is about 3000 km. If the light spends more than 1/100th of a sec. in the interferometer, the changes in distance would cancel out while the light was in the interferometer.

The interferometers are about

about 1/40000 of a second going back and forth. They design them so that the light bounces back and forth about ~~100~~ 1000 times so it spends about 1/40 of a period travelling back and forth. each time the wave in one arm takes a tiny fraction (10^{-22}) longer in one arm than the other. When the two waves come back together, one is a tiny fraction of a wavelength or period later. and the interference at the beam splitter varying the intensity of light coming out the various sides of the beam splitter. By measuring those variations one can tell how much longer one path was than the other.

Of course there are lots of noise sources. If the ground at one of the masses shakes, it will move that mass and not the other, which might mask the grav. wave signal. If wind blows turbulently the density of the air above one mass might be greater than over the other, and the grav. effect of the different masses might move one mass more than the other. The current detectors are so sensitive they can detect the vibrations caused by ocean waves crashing on the beach 1000 km away.

All of these "classical" noise sources have been knocked down so much by isolating the detectors from outside disturbances that there is a fundamental noise source – the quantum nature of the light in the interferometer – that is now the most important noise source.

There are two quantum effects which turn out to be complementary. One is called "shot noise". When the two beams interfere at the beam splitter, the E.M radiation is detected by "particle detectors" – photon detectors – While the average number of photons is proportional to the classical E.M. intensity, the number of photons fluctuates even if the beam has the same intensity with a Poisson distribution. These fluctuations look like they are signals and thus mask the real signal.

The other fluctuation is caused by the light hitting the mirrors Just as the light detected has shot noise fluctuations, so does the light pressure hitting the mirror.

These forces cause the mirror to move, which of course causes fluctuations in the output, and mask the gravity wave signal.

By increasing the intensity of the light, the fractional effect on the output due to output shot noise gets smaller fractionally, so this would suggest making the laser more and more powerful. But it is the fluctuations (not the fractional fluctuations) of the forces on the mirrors that are important. Thus a powerful laser makes the noise due to fluctuating radiation pressure larger and larger.

Both of these noise sources arise from the quantum nature of the laser beam. (When they interact with the masses or the detector they act like particles, not continuous E.M. Radiation).

So, let us look at the system. To simplify, let us assume that the system along each arm of the interferometer is 1 dimensional (i.e. neglect the problems that the beam is of finite width, and suffers from diffraction, etc.)

~~Now~~ Label that one dimension as x and y. Thus the equation for one of the polarizations is a scalar field, massless.

$$\mathcal{L} = \frac{1}{4}\int (\partial_t \phi(t,x))^2 + (\partial_x \phi(t,x))^2 \, dx$$

At the end of the interferometer

~~one case~~

The other arm has

$$\mathcal{L} = \frac{1}{2}\int \left((\partial_t \phi)^2 - (\partial_y \phi)^2 \right) dy .$$

At the end of the interferometer we have a suspended mirror in each case with mass M and spring constant k so that

$$\mathcal{L} = \frac{M}{2\cancel{M}}\left[\left(\frac{dX}{dt}\right)^2 - \Omega^2 X^2\right] \quad \text{and}$$

$$\mathcal{L}_2 = \frac{M}{2M}\left[\frac{1}{\cancel{M}}\left(\frac{dY}{dt}\right)^2 - \Omega^2 Y^2\right] \qquad \left(\frac{M\Omega^2}{\cancel{M}} = k\right)$$

Let us assume that Ω and M are identical for both mirrors.

The boundary condition (interaction between the mirror and the field)

is $\phi(t, X+L) = 0$; $\psi(t, Y+L) = 0$
(L is the distance from the half silvered
This is of course a highly non linear interaction, but we will assume that $X = X_0 + \delta X$,

$Y = Y_0 + \delta Y$ where Y_0 and X_0 are assumed to be constants.

Similarly, we assume that

$\phi = \phi_0 + \delta\phi \qquad \psi = \psi_0 + \delta\psi.$

where ϕ_0, ψ_0 obey $\phi_0(t, L+X_0) = 0$

$\psi_0(t, L+Y_0) = 0.$

~~Plugging into the~~

Furthermore, $\psi = \psi_0 + \delta\psi$

We have

$\mathcal{L} \triangleq \int \psi_0$

The equations of motion are

$$\partial_t^2 \psi - \partial_x^2 \psi = 0$$
$$\partial_t^2 Y - \Omega^2 Y = 0$$

$$\int^{Y} \left(\frac{\partial \psi}{\partial t}\right)^2 - \left(\frac{\partial \psi}{\partial y}\right)^2 dy + \frac{M}{2}\left(\dot{Y}^2 + \Omega^2 Y\right)$$

$$\left(\frac{\partial \psi_0}{\partial t}\right)^2 - \left(\frac{\partial \psi_0}{\partial y}\right)^2$$

$$+ \int^{Y_0 + \delta y} \frac{\partial \psi_0}{\partial t}\frac{\partial \delta \psi}{\partial t} - \frac{\partial \delta \psi}{\partial x}\frac{\partial \delta \psi}{\partial x} + \frac{\partial Y}{\partial t}\,\partial Y_0$$

$$= \frac{\partial \psi_0}{\partial x}^2 (Y_0 + \delta Y)$$

$\partial_x \psi \, \delta \psi$

$$\frac{\partial \psi_0}{\partial x}(L + Y_0) +$$

$$m\left[\frac{\partial^2 \psi}{\partial t^2} + \Omega^2 Y\right] + \left(\frac{\partial \psi}{\partial x}\right)^2$$

$$\frac{\partial \psi_0}{\partial x}\frac{\partial \delta \psi}{\partial x} = \delta \ddot{Y} + \Omega^2 \delta Y$$

$\partial_x \psi_0 \; \delta Y \, \partial_y \delta \psi \Big|_y$

$\delta \psi(t+x) + \delta \psi(t-x)$

$$\Box \delta \psi = \left(\frac{\partial \psi}{\partial x}\right)^2 \delta Y$$

$$\ddot{Y} + \Omega^2 Y$$

$$\delta \ddot{Y} + \Omega^2 \delta Y = \frac{\partial \psi}{\partial x}\frac{\partial \delta \psi}{\partial x}$$

$$\mathcal{L} = \frac{1}{2}\left\{\int_0^{L+Y}\left[(\partial_t \psi)^2 - (\partial_y \psi)^2\right]dx + m\left[(\partial_t Y)^2 - \Omega^2 Y^2\right]\right.$$

$$\psi \longrightarrow \quad \partial_t^2 \psi - \partial_y^2 \psi = 0$$

$$\psi = \psi_R(t-y) + \tilde{\psi}_L(t+y)$$

Boundary cond.

$$\psi_R(t-(L+Y(t))) = -\tilde{\psi}_L(t+L+Y(t))$$

$$\tilde{\psi}_L(t) = -\psi_R(t-2(L+Y(t)))$$

Varying Y.

$$m\,\partial_t^2 Y + \Omega^2 Y - \frac{1}{2}\left[(\partial_t \psi)^2 - (\partial_x \psi)^2\right]_{y=L+Y}$$

$$\partial_t \psi\Big|_{y=L+Y} = 0$$

$$m(\partial_t^2 Y + \Omega Y) = -\frac{1}{2}(\partial_x \psi)^2\Big|_{y=L+Y}$$

Set $\psi = \psi_0 + (\,)\delta Y + \delta\psi$

$Y = Y_0 + \delta Y$

where Ψ_0 is large and lets say is a single frequency. It is a "classical" field which obeys the field equations. Y_0 will be the effect on Y of the classical field Ψ_0. ~~and~~ so that to lowest order. Because the freq of Ψ_0 is so much higher than Ω

$$\frac{1}{2} m \Omega^2 Y_0 = -\frac{1}{2} (\partial_x \Psi_0)^2 \Big|_{y=L+Y_0}$$

~~Assume~~

$$\cancel{\Psi_0 = a \cos\omega t \sin(\omega(y-L-\Psi_0))}$$

will be the 0^{th} order solution. a is the amplitude.

$$\partial_x \Psi_0 \Big|_{y=L+Y_0} = \omega a \cos\omega t \cos\omega(y-L-Y_0)$$

$$= \omega a \cos\omega t$$

$$(\partial_x \Psi_0)^2 \approx \frac{1}{2} \omega^2 a^2$$

$$\psi_R(t-y) = \psi_{R0}(t-y) + \delta\psi_R(t-y)$$

$$\psi_L(t+y) = \psi_{L0}(t+y) + \delta\psi_L(t+y)$$

$$\psi_L(t+y) = -\psi_R(t+y-2(L+Y_0+\delta Y))$$

$$= -\psi_{R0}(t+y-2(L+Y_0)) - \delta\psi_1(t+y-2(L+Y_0))$$

$$+ \frac{\partial\psi_{R0}}{\partial y}\left(t+y-2(L+Y_0)\right)\delta Y\, 2$$

(we assume δY and $\delta\psi$ are same order)

$$\psi_R = \cancel{\cos\omega t\,\cos(\omega y)}\; a\cos\omega(t-y) + \delta\psi_R$$

$$\psi_L = -a\cos(\omega(t+y-2(L+Y_0))) - \delta\psi_R(t+y-2(L+Y_0))$$
$$+ a\omega\sin(\omega(t+y-2(L+Y_0)))\, 2\delta Y.$$

$$\psi = \cancel{a(\sin(\omega(t-L)))}$$

$$= 2a(\sin(\omega(t-L-Y_0))\sin(\omega(y-L-Y_0)))$$

$$+ \delta\psi_R((t-L-Y_0)-(y-L-Y_0)) - \delta\psi((t-L-Y_0)$$
$$+(y-L-Y_0))$$

$$+ a\omega\sin(\omega(t+y-2(L+Y_0))\, 2\delta Y($$

Incoming $\psi_R(t-y)$
Outgoing $\psi_L(t+y)$

$$\psi_R(\underbrace{t-\hat{L}-\delta Y(t)}) = -\psi_L(\underbrace{t+\hat{L}+\delta Y(t)}_{\tau})$$

$$\psi_L(\tau) = -\psi_R(\tau - 2(L+\delta Y(\tau - \hat{L})))$$

$$\psi = \psi_R(t-y) - \psi_R(t+y-2(L+\delta Y(t+y-\hat{L})))$$

$$\frac{1}{2}\left(\frac{\partial\psi}{\partial y}\right)^2\Bigg|_{y=\tilde{L}+\delta Y} = \frac{1}{2}\left[-\dot{\psi}_R - \dot{\psi}_R(t-\tilde{L}-\delta Y(t))\,[1-2\delta\dot{Y}(t)]\right]^2$$

$$= \left[-\cancel{2}\,\dot{\psi}_R(t-L-\delta Y)[\cancel{2}-2\delta\dot{Y}(t)]\right]^2$$

= deriv of argument.

$$\simeq \frac{1}{2}\left\{\cancel{4}\,\dot{\psi}(t-L-\delta Y)^2 - \cancel{8}\,\dot{\psi}^2(t-L)\,\delta\dot{Y}\right.$$

$$= \cancel{\tfrac{1}{2}}\,2\,\dot{\psi}_R(t-L)^2 - \underbrace{\dot{\psi}\ddot{\psi}\,\cancel{4}\,\delta Y}_{\text{freq } 2\omega e^{\pm 2i\omega t}} - \cancel{4}\,\dot{\psi}^2\,\delta\dot{Y}$$

$$\simeq \underbrace{2\dot{\psi}_R(t-L)^2}_{\text{constpart}}(1-2\delta\dot{Y})$$

Thus.

$$m\left(\partial_t^2 Y + \Omega^2 Y\right) = \left(\partial_y \Psi_0\right)^2 (1 - 2\dot{Y})$$

The second term on the RHS is a damping term for the oscillator.

The pressure on the mirror due to the radiation ~~for~~ bouncing off the mirror is just equal to the energy density at the mirror. (½ the energy comes toward the mirror with momentum equal to the energy density (with $c=1$), and half travels away with the opposite ~~energy~~ momentum density. Thus the change in momentum is $2 \times (\frac{1}{2}$ energy density$)$ = energy density.)

this is $\frac{1}{2}\left[\left(\frac{\partial \psi}{\partial t}\right)^2 + \left(\frac{\partial \psi}{\partial y}\right)^2\right]$

At the mirror the ~~energy density~~ $\Psi = 0$ or $\frac{\partial \Psi}{\partial t} = 0$ as well.

This the pressure is $\frac{1}{2}\left(\partial_y \Psi\right)^2$

Thus The Harmonic osc eqn is

$$\left(\partial_t^2 Y\right) + \Omega^2 Y = -\frac{1}{2}\left(\frac{\partial \psi}{\partial y}\right)^2\Bigg|_{y=\tilde{L}+Y}$$

$$\Psi_{0R} = a\cos(t-y)$$

$$\psi = \Psi_{0R}(t-y) - \Psi_{0R}\left(t+y-2(\tilde{L}+Y(t+y-\tilde{L}))\right)$$

$$+\ \delta\Psi(t-y) - \delta\Psi(t+y-2\tilde{L})$$

$$\cancel{\frac{1}{2}\left(\frac{\partial \psi}{\partial y}\right)^2 = \frac{1}{2}\Big[}$$

$$\cancel{\frac{a^2}{2}\left(\partial_y\left(\cos(t-y) - \cos(t+y-2(\tilde{L}+Y(t+y-\tilde{L}))\right)\right)^2_2}$$

$$\cancel{\left(-\sin(t-(\tilde{L}+Y(t)) - \sin(t-\tilde{L}+Y(t))(1-2\dot{Y}(t)\right)}$$

$$\cancel{\approx \frac{1}{2}\left(4\sin^2(t-(L+Y(t))\,(1-4\dot{Y})\right)}$$

$$\psi_{0R} = a\cos\omega(t-y)$$

$$\frac{1}{2}\left(\frac{\partial\psi_0}{\partial y}\right)^2 = \frac{1}{2}4\omega^2a^2\sin^2(\omega(t-\tilde{L}))(1-\dot{Y})$$

$$\text{Pressure} = 2\omega^2a^2\sin^2(\omega(t-\tilde{L}))(1-\dot{Y})$$

$$\approx \omega^2a^2(1-\dot{Y})$$

since the terms which go as $e^{\pm i\omega 2t}$ average out to zero over very short times $> 1/\omega$.

This

$$\partial_t^2 Y + \frac{\omega^2a^2}{m}\partial_t Y + \Omega^2 Y = \frac{\omega^2a^2}{m}.$$

~~for~~ If we neglect the $\delta\psi$ term.

This eqn gives

$$Y_0 = \frac{\omega^2}{\Omega^2}\frac{a^2}{m}$$ as a constant displacement due to the radn pressure

$$\partial_y \delta\Psi = 2\partial_t \delta\Psi(t-L)$$

Keeping only 1st order in $\delta\Psi$ we get.

$$\partial_t^2 \delta\Psi + \frac{\omega^2 a^2}{m} \partial_t \delta\Psi + \Omega^2 \delta\Psi =$$

$$-\frac{\lambda}{2}\left(\partial_y \Psi_0 \, \partial_y \delta\Psi\right)$$

$$= -2\omega a \sin\omega(t-L)\left(-\partial_t \delta\Psi_{0R}\right)$$

$$= +2\omega a \sin(\omega(t-\tilde{L}))\, \partial_t \delta\Psi_{0R}(t-\tilde{L})$$

This will only pick up the fluctuations with frequency near $\pm\,\omega$.

$\delta\Psi$ will be a quantum field

(if Ψ is a quantum field then $\Psi - \Psi_0$ has exactly the same commutation relns and eqn of motion)

$$\delta\psi_R = \int\left(A_{\tilde{\omega}}\,\frac{e^{-i\tilde{\omega}(t-y)}}{\sqrt{4\pi\tilde{\omega}}} + A^{+}_{\tilde{\omega}}\,\frac{e^{i\tilde{\omega}t-y}}{\sqrt{2\pi\tilde{\omega}}}\right)d\omega$$

$$P_t = 2a\omega \sin\omega(t-\tilde{t})\,\partial_t\delta\psi$$

$$= 2a\omega\Big(e^{i\omega(t-\tilde{t})}\,A_{\tilde{\omega}}\frac{-i\tilde{\omega}}{2i\sqrt{4\pi\tilde{\omega}}}\,e^{-i\tilde{\omega}t-y)}$$

$$\Big(-\,e^{-i\omega(t-t)}\,A^{+}_{\tilde{\omega}}\frac{+i\tilde{\omega}}{2i\sqrt{4\pi\tilde{\omega}}}\,e^{i\tilde{\omega}(t-y)}\Big)$$

$$= \frac{2\omega a}{2i}\Big(e^{-i(\tilde{\omega}-\omega)(t-t)}\,A_{\tilde{\omega}}\sqrt{\frac{\tilde{\omega}}{4\pi}}$$

$$-\,e^{+i(\tilde{\omega}-\omega)(t-t')}\,A^{+}_{\tilde{\omega}}\sqrt{\frac{\tilde{\omega}}{4\pi}}\Big)$$

~~δψ(ω̂)~~

$$\delta Y(\hat{\omega}) = \frac{A_{\omega+\hat{\omega}} + A^{+}_{(\omega+\hat{\omega})}}{-\hat{\omega}^2 - i\hat{\omega} + \Omega^2}$$

$$\delta\psi_R = \int\left(A_{\tilde{\omega}} \frac{e^{-i\tilde{\omega}(t-y)}}{\sqrt{4\pi\tilde{\omega}}} + A^+_{\tilde{\omega}} \frac{e^{i\tilde{\omega}t-y}}{\sqrt{2\pi\tilde{\omega}}}\right)d\omega$$

$$P_t = 2a\omega \sin\omega(t-\tilde{t})\partial_t\delta\psi$$

$$= 2a\omega\Big(e^{i\omega(t-\tilde{t})}\, A_{\tilde{\omega}} \frac{-i\tilde{\omega}}{2i\sqrt{4\pi\tilde{\omega}}}\, e^{-i\tilde{\omega}t-y)}$$

$$\Big(-e^{-i\omega(t-t)}(A^+_{\tilde{\omega}} \frac{+i\tilde{\omega}}{2i\sqrt{4\pi\tilde{\omega}}}\, e^{i\tilde{\omega}(t-y)}\Big)$$

$$= \frac{2\omega a}{2i}\Big(e^{-i(\tilde{\omega}-\omega)(t-t)}\, A_{\tilde{\omega}}\sqrt{\frac{\tilde{\omega}}{4\pi}}$$

$$- e^{+i(\tilde{\omega}-\omega)(t-t')}\, A^+_{\tilde{\omega}}\sqrt{\frac{\tilde{\omega}}{4\pi}}\Big)$$

~~$\delta Y(\omega)$~~

$$\delta Y(\hat{\omega}) = \frac{A(\omega+\hat{\omega}) + A^+(\omega+\hat{\omega})}{-\hat{\omega}^2 - i\hat{\omega} + \Omega^2}$$