Physics 501
Final Exam
Apr 20-21 24 hour Takehome exam

By submitting this exam you affirm that you have talked to no-one else about this exam, and received help from no-one else in answering the questions in this exam before you emailed the exam to

unruh@physics.ubc.ca

Please email the answers

by 3:30PM, Apr 21 2020 PDT

If you have any trouble submitting this exam, please immediately call, before 3:30 PM, Apr 21 2020 PDT

W Unruh 604 736 5745 or 778 238 7962

Answer five (5) of the following questions. If you answer more than 5, let me know which 5 answers you want me to mark. If you do not tell me, I will simply mark the first 5 that you answer. All questions are worth the same number of marks (5 points)

1) [Adiabatic]Consider the two-level system with Hamiltonian

$$H = \cos(\theta)\sigma_z + \sin(\theta)\sigma_x \quad (1)$$

$$\theta(t) = \frac{t}{T} \quad (2)$$

where T is some constant. Define the states $|1\rangle$ and $|0\rangle$ by $\sigma_z |0\rangle = -|0\rangle$ and $\sigma_z |1\rangle = |1\rangle$.

a)What are the eigenstates and eigenvalues of $H(t)$.

b) At $t = 0$ assume that the initial state of the system is the lowest energy eigenstate of $H(0)$. If $T << 1$, what will be the state of the system at time $t = \pi T$?

c) At $t = 0$ assume that the intial state of the system is in the lowest energy eigenstate of $H(0)$. Assume that $T >> 1$. What will be the probability that the state will still be the ground state at $t = \pi T$?

Let the state be $|\phi\rangle = \alpha(t)|0\rangle + \beta(t)|1\rangle$. What are the Schroedinger equations for α and β?

Show that the equations for $\alpha + i\beta$ and $\alpha - i\beta$ are

$$\partial_t(\alpha + i\beta) = ie^{-i\theta(t)}(\alpha - i\beta) \tag{3}$$
$$\partial_t(\alpha - i\beta) = ie^{i\theta(t)}(\alpha + i\beta) \tag{4}$$

(note that both α and β will in general be complex).
Define

$$Z_1 = e^{i\theta(t)/2}(\alpha + i\beta) \tag{5}$$
$$Z_2 = e^{-i\theta(t)/2}(\alpha - i\beta) \tag{6}$$

Find the first order equations obeyed by Z_1, Z_2 and show that they both obey the same second-order differential equation.
Given the initial conditions on α and β, assuming that the intial state is the lowest eigenstate of $H(0)$, find what α and β are at time $t = \pi T$. What is the lowest energy state at $t = \pi T$ and what is the probability that the system is still in its ground state at time $t = \pi T$?
Note that both α and β are complex, so that $\alpha + i\beta$ is not the complex conjugate of $\alpha - i\beta$.
(This is a poor-man's version of the Landau-Zenner proof).

2.) [Cosmology] Consider a massive scalar field in a deSitter cosmology where $a(t) = e^t$ and in 1+1 dimensions (ie only one spatial dimension) . The Lagrangian for a scalar field in this spacetime is

$$\mathcal{L} = \frac{1}{2}\int (a(t)(\partial_t\phi)^2 - \frac{1}{a(t)}(\partial_x\phi)^2) + m^2 a(t)\phi^2 \tag{7}$$

a)Find the (time-dependent) Hamiltonian. Doing one round of the adiabatic transformation, what are $\hat{\phi}$ and $\hat{\pi}$ and τ in terms of ϕ, π, t (as defined in the notes)?

b) Under what conditions would you expect this to be a good approximation?

3) [Balancing pencil] Consider a pencil, length L, mass m, and with the moment of inertia about the point being $I = mL^2/3$. The pencil is balanced verticall on its point, with θ being the angle from the vertical. I model the Hamiltonian by

$$H = \frac{1}{2}(\frac{3p^2}{mL^2} - \frac{mgL}{2}\theta^2) \tag{8}$$

where p is the momentum conjugate to θ.
a)Find the relation between $\hat{p}$ and $\hat{\theta}$ and p and θ such that the Hamiltonian becomes

$$H = \frac{\Omega}{2}(\hat{p}^2 - \hat{\theta}^2) \tag{9}$$

P and Θ are the associated quantum operators. P and Θ are the quantum operators associated with p and θ.
Choosing a solution $\hat{\theta} = 1/\sqrt{2}(\alpha e^{\Omega t} + i\beta e^{-\Omega t})$, find the value β as a function of α that makes this a unit postive norm solution. Define the annihilation operator A_α corresponding to this solution.
Choose the vacuum state defined by

$$A_\alpha \left|\alpha\right\rangle = 0 \tag{10}$$

What value of α will maximize the time such that

$$\left\langle\alpha\right| \Theta(t)^2 \left|\alpha\right\rangle = 1 \quad ? \tag{11}$$

(Note that $\left\langle\alpha\right| \Theta(t)^2 \left|\alpha\right\rangle$ must be less than 1 at t=0.)
Use $\hbar = 10^{-34} kg - m^2/s$, $m = .01kg$, $g = 10m/s^2$, $L = .1m$.
For $\alpha = 1$, and $t = 0$ choose the state

$$\left|\phi\right\rangle = cos(\phi) \left|\alpha\right\rangle + sin(\phi)\frac{1}{\sqrt{2}} A_\alpha^{\dagger 2} \left|\alpha\right\rangle \tag{12}$$

Are there values of ϕ which make the expectation value of $\left\langle\phi\right| Q(0)^2 \left|\phi\right\rangle$ smaller than when $\phi = 0$?

4) [quantum foundation] Describe in some detail the Hardy chain and its implications for the possibility of finding a classical description of a quantum system.

5) Given two orthogonal positive norm modes of a linear system, with annihilation operators $A,\ B$ and two other orthogonal positive norm modes with annihilation operators $C,\ D$, and given that the relation between $A,\ B$ and $C,\ D$ is.

$$C = \alpha A + \beta B^\dagger \tag{13}$$
$$D = \gamma B + \delta A^\dagger \tag{14}$$

a)What are the relations between $\alpha,\ \beta,\ \gamma,\ \delta$ so that $C,\ D$ obey the annihilation/creation operator commutation relations?
b) Given the vacuum state for A and B ($A\left|00\right\rangle = B\left|00\right\rangle = 0$) what is the expectation value for number operator $C^\dagger C + D^\dagger D$ and for the difference operator $C^\dagger C - D^\dagger D$ in this state?
c) Assume $\alpha,\ \beta,\ \gamma,\ \delta$ are all real. Express the vacuum state of the $C,\ D$ modes in terms of the vacuum state of the $A,\ B$ modes and appropriate operators acting on it.

6)[Quantumfoundations]. Alice and Bob each have a two-level system with eigenstates $\left|0\right\rangle_A,\ \left|1\right\rangle_A$ and $\left|0\right\rangle_B,\ \left|1\right\rangle_B$. They arrange that the state of the joint system is one of the Bell states $(\left|0\right\rangle_A \left|1\right\rangle_B - \left|1\right\rangle_A \left|0\right\rangle_B)/\sqrt{2}$.Charlie now wanders up with his state $\alpha\left|0\right\rangle_C + \beta\left|1\right\rangle_C$ with $|\alpha|^2 + |\beta|^2 = 1$. Alice takes Charlie's system in that unknown state together with her own system and measures that the joint system is one of the four Bell states

$$(\left|0\right\rangle_A \left|1\right\rangle_C - \left|1\right\rangle_A \left|0\right\rangle_C)/\sqrt{2} \tag{15}$$
$$(\left|0\right\rangle_A \left|1\right\rangle_C + \left|1\right\rangle_A \left|0\right\rangle_C)/\sqrt{2} \tag{16}$$
$$(\left|0\right\rangle_A \left|0\right\rangle_C - \left|1\right\rangle_A \left|1\right\rangle_C)/\sqrt{2} \tag{17}$$
$$(\left|0\right\rangle_A \left|0\right\rangle_C + \left|1\right\rangle_A \left|1\right\rangle_C)/\sqrt{2} \tag{18}$$

and sends the result of that measurement to Bob (Ie, she tells Bob which of the four results she got).

a)Show that the 4 states are the eigenvectors with distinct eigenvalues of

$$BS = 2\sigma_{zA}\sigma_{zC} + \sigma_{xA}\sigma_{xC}. \tag{19}$$

Depending on the result obtained by Alice, Bob now carries out a Unitary transformation on his own two-level system.

b) Show that the probability of Alice getting any one of the four Bell states when she measures BS is independent of α and β and is the same for each possible outcome.
c)Depending on which of the four results Alice got, what is the Unitary transformation Bob needs to carry out on his system so that he is certain he now has the same state as Charlie had originally (which of course he has no idea what it is). Ie, he wants to be sure that he has the state $\alpha \left|0\right\rangle_B + \beta \left|1\right\rangle_B$.

7) Make up your own question on any of the material discussed in this course and answer it. The question will be marked both on the quality of the question and of the answer. A trivial question with the correct answer will not receive high marks. The question also cannot be very similar to any of the other questions on this exam. None of the questions in this exam would be considered as trivial.
The purpose of this question is to give you the opportunity to show me that you have mastered some aspect of this course which are not covered by the questions I have asked. The only requirement is that the question have something to do with the course.

Please read and sign the next page and include an image with your exam solutions.

I confirm that I have not communicated in any form or way with anyone else about this exam or about any questions on this exam during the period assigned for the writing of this exam

Signature