Physics 501-21
Assignment 3

1) Schroedinger from Heisenberg.
a) Show that $e^{-i\epsilon P}$ is a translation operator for x.

$$e^{i\epsilon P}|x\rangle = |x-\epsilon\rangle \tag{1}$$

Thus argue that formally this implies that

$$P|x>= i\partial_x|x> \tag{2}$$

Thus for a state $|\psi\rangle$ that if $\psi(x) = \langle x| |\psi\rangle$, then

$$\langle x| P |\psi\rangle = -i\partial_x\psi(x) \tag{3}$$

2) Consider a system of an inverted Harmonic oscillator

$$H = \frac{1}{2}(p^2 - x^2) \tag{4}$$

Show that

$$x = \cosh(t) + i\sinh(t), \tag{5}$$

together with its conjugate momentum solution $p(t)$, is a positive norm solution, and thus can be used to define an annihilation operator A, and a vacuum state $|0\rangle$ for this problem.

Find the evolution of this state in the Schroedinger representation. At any time t, define $|x\rangle$ by $X(t)|x\rangle = x|x\rangle$. Use the definiton of the state $|0\rangle$ in terms of A to find state $\psi(x,t) = \langle x| |0\rangle$ at any time t. Then $\langle x| P(t) |0\rangle = -i\partial_x\psi(x,t)$ Use these and the definition of A to find the time evolution of $\psi(t,x)$, the Schroedinger wave function. (Note that this is much easier to do than to solve the Schroedinger equation directly).

2) Show the $U = e^{ir(XP+PX)/2}$ is a Unitary squeezing operatore such that

$$UXU^\dagger = e^r X \tag{6}$$
$$UPU^\dagger = e^{-r} P \tag{7}$$

(Hint, Use the operator $(PX + XP)/2$ Hamiltonian and r as the time. Show that the hamiltonian is Hermitian. Find and solve the equations of motion for X and P and the Unitary transformation for this Hamiltonian.)

3) Consider the field theory with Lagrangian

$$\mathcal{L} = \int (\partial_t\phi - v\partial_x\phi)^2 - (\partial_x\phi)^2 - (\partial_x^2\phi)^2 dx \tag{8}$$

where x represents the one dimensional coordinate.

i)Find the Hamiltonian.

ii) Assume $v > 1$. Find the modes which diagonalize this Hamiltonian. (Hint use the spatial Fourier transform), and assume that the temporal solution goes as $e^-i\omega t$. What are the solutions for ω as a function of k the wavenumber? What is the condition on ω that the solutions have positive norm?

This is the field theory which represents sound waves in a BEC, where the spatial coordinates are expressed in terms of the "healing length" and the time is chosen so that the velocity of sound at long wavelengths is unity. This is looking at the BEC in a frame moving with velocity v with respect to the rest frame of the BEC.

4.) Consider a Harmonic oscillator with configurationa and momentum q, p, and with Hamiltonian

$$H = -\frac{1}{2}(p^2 + \Omega^2 q^2) \tag{9}$$

(Yes, that sign is correct). What are the equations of motion for p and q. Assume that the mode solution goes as $e^-i\omega t$. What is ω? What is the norm of this solution?

Define the annihilation operator as usual as

$$A = \langle q, Q \rangle. \tag{10}$$

What is the Hamiltonian in terms of A, $A^\dagger$? Consider the state $|0\rangle$ defined as $A|0\rangle = 0$. What is the energy of the first excited mode $A^\dagger|0\rangle$? What can you say about the energy of $|0\rangle$ in terms of the other states of the system?