Physics 501-21
Assignment 4

1) Schroedinger from Heisenberg.
a) Show that $e^{-i\epsilon P}$ is a translation operator for x.

$$e^{i\epsilon P}\left|x\right\rangle = \left|x-\epsilon\right\rangle \tag{1}$$

Thus argue that formally this implies that

$$P|x>= i\partial_x|x> \tag{2}$$

Thus for a state $\left|\psi\right\rangle$ that if $\psi(x)=\left\langle x\right|\left|\psi\right\rangle$, then

$$\left\langle x\right| P\left|\psi\right\rangle = -i\partial_x\psi(x) \tag{3}$$

$[-iP,X]$= -1. Thus

$$[X,e^{i\epsilon P}]=\sum_{n=0}^{\infty}[X,\frac{(i\epsilon P)^n}{n!}]=\sum_n\frac{1}{n!}sum_r(i\epsilon P)^r[X,i\epsilon P](i\epsilon P)^{n-r-1} \tag{4}$$

$$=-\epsilon\sum_n\frac{1}{n!}n(i\epsilon P)^{n-1}=-\epsilon e^{i\epsilon P} \tag{5}$$

$$X(e^{i\epsilon P}\left|x\right\rangle=[X,e^{i\epsilon}]\left|x\right\rangle+e^{i\epsilon P}X\left|x\right\rangle=(x-\epsilon)e^{i\epsilon P}\left|x\right\rangle \tag{6}$$

so $e^{i\epsilon P}\left|x\right\rangle=\left|x-\epsilon\right\rangle$, the $x-\epsilon$ eigenvector of X.
Then

$$e^{i\epsilon P}\left|\psi\right\rangle=e^{i\epsilon P}\int\psi(x)\left|x\right\rangle dx=\int\psi(x)\left|x-\epsilon\right\rangle dx=\int\psi(x+\epsilon)\left|x\right\rangle dx. \tag{7}$$

so

$$\lim_{\epsilon\to0}\frac{e^{i\epsilon P}-1}{\epsilon}\left|\psi\right\rangle=\lim_{\epsilon\to0}\frac{(\psi(x+\epsilon)-\phi(x))}{\epsilon}\left|x\right\rangle \tag{8}$$

$$=\partial_x\psi(x)\left|x\right\rangle \tag{9}$$

But

$$\lim_{\epsilon\to0}\frac{e^{i\epsilon P}-1}{\epsilon}=\lim_{\epsilon\to0}\sum_{n=1}\epsilon^{n-1}\frac{(iP)^n}{n!}=iP \tag{10}$$

so

$$iP\psi(x)\int\psi(x)\left|x\right\rangle dx=\int\partial_x\psi(x)\left|x\right\rangle dx \tag{11}$$

$$\int P\psi(x)\psi(x)\left|x\right\rangle dx=\int -i\partial_x\psi\left|x\right\rangle dx \tag{12}$$

OR

Regard ϵ as time, with P as the Hamiltonian. Then the unitary matrix of evolution is

$$U = e^{-i\epsilon P} \tag{13}$$

and the Heisenberg equations of motion are

$$i\partial_\epsilon X = [X, H] = [X, P] = i \tag{14}$$
$$i\partial_\epsilon P = 0 \tag{15}$$

which has solution

$$X = X_0 + \epsilon \tag{16}$$

But also

$$X(\epsilon) = e^{i\epsilon P} X e^{-i\epsilon P} \tag{17}$$

which was to be shown. The equivalent Schroedinger equation is

$$i\partial_\epsilon \phi(x, \epsilon) = H\phi(x, \epsilon)\phi(x, \epsilon) = i\partial_x \phi(x, \epsilon)) \tag{18}$$

which has as solution $\phi(x + \epsilon)$ as can be seen by substition in the equation.
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2) Consider a system of an inverted Harmonic oscillator

$$H = \frac{1}{2}(p^2 - x^2) \tag{19}$$

Show that

$$x = \cosh(t) + i \sinh(t), \tag{20}$$

together with its conjugate momentum solution $p(t)$, is a positive norm solution, and thus can be used to define an annihilation operator A, and a vacuum state $|0\rangle$ for this problem.

——————— (Sorry, it is actually negative norm)

The equations of motion are

$$\partial_t x = p; \qquad \partial_t p = x \tag{21}$$

so $p = \sinh(t) + i\cosh(t)$ and $\partial_t p = \cosh(t) + i\sinh(t) = x$ and it is a solution. The norm is

$$< x, x >= \frac{i}{2}(x^* p - p^* x) = \frac{i}{2}((cosh(t) - i\sinh(t))(sinh(t) + i\cosh(t) - (sinh(t) - icosh(t))(cosh(t) + i\sinh$$

so this is a negative norm solution. The positive norm is the complex conjugate. Thus

$$A = \frac{i}{2}((cosh(t) - i\sinh(t))^* P(t) - (sinh(t) - i\cosh(t))^* X(t)) \tag{23}$$

Now defining x as the eigenvalue of $X(t)$, and $P(t) = -i\partial_x$, we have

$$0 = A\psi(x,t) = \frac{i}{2)((cosh(t) + i\sinh(t))(-i\partial_x) - (sinh(t) + i\cosh(t))x\psi(x,t))} \tag{24}$$

$$= \frac{1}{2}(cosh(t)_i \sinh(t))\left(\partial_x - i\frac{sinh(t) + icosh(t)}{cosh(t) + isinh(t)}\right)\psi(x,t) \tag{25}$$

or

$$\psi(x,t) = exp(i\frac{1}{2}\frac{sinh(t) + i\cosh(t)}{\cosh(t) + i\sinh(t)}x^2) \tag{26}$$

$$\psi(x,t) = exp(-\frac{1}{2}\frac{1 - i\sinh(2t)}{\cosh(2t)}x^2) \tag{27}$$

This gives us the solution to the evolution of a gaussian inital wave packet far more easily that trying to solve the Schroedinger equation directly.

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Find the evolution of this state in the Schroedinger representation. At any time t, define $|x\rangle$ by $X(t)|x\rangle = x|x\rangle$. Use the definiton of the state $|0\rangle$ in terms of A to find state $\psi(x,t) = \langle x| |0\rangle$ at any time t. Then $\langle x| P(t) |0\rangle = -i\partial_x\psi(x,t)$ Use these and the definition of A to find the time evolution of $\psi(t,x)$, the Schroedinger wave function. (Note that this is much easier to do than to solve the Schroedinger equation directly).

2– actually 3) Show the $U = e^{ir(XP+PX)/2}$ is a Unitary squeezing operatore such that

$$UXU^\dagger = e^r X \tag{28}$$

$$UPU^\dagger = e^{-r} P \tag{29}$$

(Hint, Use the operator $(PX + XP)/2$ Hamiltonian and r as the time. Show that the hamiltonian is Hermitian. Find and solve the equations of motion for X and P and the Unitary transformation for this Hamiltonian.)

This is obviously problem 3

The equations of motion for our H and r as time is

$$\partial_r X = X\partial_r P = -P \tag{30}$$

so the solutions are

$$X(r) = e^r X(0) \tag{31}$$

$$P(r) = e^{-r} P(0) \tag{32}$$

But

$$X(r) = e^{irH} X e^{-irH} \tag{33}$$

and similarly for $P(r)$.

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3– should be 4) Consider the field theory with Lagrangian

$$\mathcal{L} = \int (\partial_t\phi - v\partial_x\phi)^2 - (\partial_x\phi)^2 - (\partial_x^2\phi)^2 dx \tag{34}$$

where x represents the one dimensional coordinate.

i)Find the Hamiltonian.

$$\pi(x) = \frac{\delta\mathcal{L}}{\delta\partial_t\phi(x)} = (\partial_t\phi - v\partial_x\phi) \tag{35}$$

$$\tag{36}$$

Then $H = \int \pi(x)\partial_t\phi(x)dx - \mathcal{L}$where we solve for $\partial_t\phi$ in terms of π. This gives

$$H = \frac{1}{2}\int (\pi(t,x)^2 - 2v\partial_x\phi(t,x)\pi(t,x)(\partial_x\phi(t,x))^2 + (\partial_x^2\phi(t,x))^2 \tag{37}$$

The equations of motion can be found either from the Lagrangian or the Hamiltonian

$$(\partial_t - \partial_x)(\partial_t - v\partial_x)\phi(t,x) - \partial_x^2\phi(t,x) + \partial_x^4\phi(t,x) = 0 \tag{38}$$

$$\pi(t,x) = \partial_t\phi - v\partial_x\phi) \tag{39}$$

or the from the Hamiltonian the first equation is

$$(\partial_t - v\partial_x)\pi(t,x) = \partial_x^2\phi(t,x) \tag{40}$$

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ii) Assume $v > 1$. Find the modes which diagonalize this Hamiltonian. (Hint use the spatial Fourier transform), and assume that the temporal solution goes as $e^-i\omega t$. What are the solutions for ω as a function of k the wavenumber? What is the condition on ω that the solutions have positive norm?

$$\phi(t,x) = \phi_{\omega k}e^{-i\omega t}e^{ikx} \tag{41}$$

$$\pi(t,x) = \pi_{\omega k}e^{-i\omega t}e^{ikx} \tag{42}$$

which gives

$$(-i\omega - ivk)^2 + k^2 + k^4 = 0 \tag{43}$$

or $(\omega+vk)^2 = k^2$ or $\omega = -vk \pm k = (-vk \pm k\sqrt{1+k^2}$. For $v > 1$, both solutions for $k > 0$ have negative ω while for $k < 0$ both are positive for small enough k. For larger $|k|$ we again get both negative and positive values of ω.

$$< \phi, \phi' >= \frac{i}{2} \int (\phi(t,x)^* \pi' t, x - \pi^*(t,x)\phi'(t,x) \tag{44}$$

$$= \frac{i}{2} \int \phi^*_{\omega k}(-i\omega' - ivk')\phi_{\omega' k'} - (i\omega + ivk)\phi^*_{\omega k}\phi_{\omega' k'})e^{-i(\omega'-\omega)t + (k-k')x} dx \tag{45}$$

$$= ((\omega + \omega')/2 - vk)(\phi^*_{\omega k}\phi'_{\omega} k' 2\pi\delta(k,k') \tag{46}$$

If we want the norm, then we $\omega' = \omega$ at the same value of k and $< \phi, \phi' >= \pm k\sqrt{1+k^2}|\phi_{\omega k}|^2$ for small positive k, even though both possible values of ω are negative, one of them has positive norm, and the other negative. The norm is NOT the same as the sign of ω in this case. For large values of k the norm is the same sign as ω.

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This is the field theory which represents sound waves in a BEC, where the spatial coordinates are expressed in terms of the "healing length" and the time is chosen so that the velocity of sound at long wavelengths is unity. This is looking at the BEC in a frame moving with velocity v with respect to the rest frame of the BEC.

4.– should be 5) Consider a Harmonic oscillator with configurationa and momentum q, p, and with Hamiltonian

$$H = -\frac{1}{2}(p^2 + \Omega^2 q^2) \tag{47}$$

(Yes, that sign is correct). What are the equations of motion for p and q. Assume that the mode solution goes as $e^- i\omega t$. What is ω? What is the norm of this solution?

$$\partial_t x = -p \tag{48}$$

$$\partial_t p = \Omega x \tag{49}$$

$$\omega = \pm\Omega \tag{50}$$

If we have $x = x_0 e^i \Omega t$, the $p = -i\Omega x$ and $< x, x >= \Omega x_0^2 > 0$. Ie, the positive norm normalised solutions go as $x = \frac{e^{i\Omega t}}{\sqrt{\Omega}}$ and $p = -i\sqrt{\Omega} e^{i\Omega t}$ rather than the usual $e^{-i|\omega|t}$. Defining

$$X = Ae^{i\Omega t} + A^\dagger e^{-i\Omega t} \tag{51}$$

$$P = i(-Ae^{i\Omega t} + A^\dagger e^{-i\Omega t} \tag{52}$$

and $H = -\frac{1}{2}\Omega(AA^\dagger + A^\dagger A)$. The vacuum state $A|0\rangle >= 0$ thus has energy $-\frac{1}{2}\Omega$ and the excited states have energy $-(n + \frac{1}{2})\Omega$. Ie the specturm gets more and more negative the more excited the oscillator is. The "vacuum" state thus has the highest energy of any of the states rather than the lowest.

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Define the annihilation operator as usual as

$$A = \langle q, Q \rangle. \tag{53}$$

What is the Hamiltonian in terms of A, $A^\dagger$? Consider the state $|0\rangle$ defined as $A|0\rangle = 0$. What is the energy of the first excited mode $A^\dagger|0\rangle$? What can you say about the energy of $|0\rangle$ in terms of the other states of the system?