Physics 501-20
Assignment 2

1.) To show the unusual power of "post-selected" quantum mechanics, Aharonov developed the "Mean King" puzzle. A Physicist is shipwrecked on an island which is ruled by a King. The king loves cats and thus hates physicists. But to be fair, he gives the physicists a problem. He has on his island a laboratory in which has all of the greatest and latest equipment. He gives this problem.

He hands the physicists a spin 1/2 particle which is in an isolated chamber such that he can assure the physicists that the free Hmiltonian is 0. Now he says that the physicists can use the lab and that spin 1/2 particle. After the physicist has done whatever he wants, the King takes that spin 1/2 particle and makes a measurement of either the spin in the x direction, the spin in the y direction, or the spin in the z direction, but does not tell the physicist which he measured. He then hands the physicist back the particle and the physicists can again do whatever he wants. The physicists is then called into the throne room, and told which spin the king measured, and the physicist is asked what the value was that was measured. This is repeated many times. If the physicist ever gets the wrong answer, she is killed. How can she survive?

Note that if there were just two possibilities, say spin in the x or y directions, it would be easy. Before hand hand she would prepare the state as an eigenstate in the x direction. Afterwards she would measure it in the y direction. If the king measured it in the x direction, the initial state would tell her what his answer would have been. If the king measured it in the y direction, the final measurement would tell her what the answer was. But with three possibilities, this clearly would not work. (If she carried out the above, and the king measured it in the z direction, there would only be a 50% chance of getting the answer) and thus she would have a good chance of dying. (1/6 probability on each repetition).

The answer is in the paper

Aharonov, Vaidman, Albert Phys Rev Letters 58, 1385 (1987)

Explicitly verify the various claims in the paper. Eg, prove that the states in eqn2 are correct and that the outcomes in the table above eqn 2 are right.

Or find a simpler example of how one would solve this connundrum.

2) Assume that we have a Hamiltonian

$$H = \frac{1}{2}\left(\frac{p_1^2}{m_1^2} + \frac{p_2^2}{m_2} + k_1x_1^2 + k_2x_2^2 + 2\epsilon x_1x_2\right) \tag{1}$$

a)What are the 4 eigenvalues $\pm i\omega_1,\ \pm\omega 2$ of the Hamiltonian equations for this Hamiltonian in terms of the constants $m_1, m_2, k_1, k_2, \epsilon$.

b) Is there any condition on $k_i, m_i, epsilon$ such that $\omega_1 = \omega 2$?

c) If $m_1 = m_2$, $k_1 = k_2$, is there any condition on ϵ such that the eigenvalues are not purely imaginary?

d) What are the normalised (using the symplectic norm) eigenvectors if $m_1 = m_2$, $k_1 = k_2$ and $\epsilon \neq 0$?

3. Consider the Hamiltonian $H = \frac{1}{2}(p^2 - x^2)$. What are the eigenvalues of the Hamiltonian ? Show that there are no purely imaginary eigenvalues.

Find a positive norm, normalised mode values of the initial momentum and position. What is the time dependence of this mode. Show that its norm is independent of time explicitly. Find the Annihilation and creation operator this mode, and show explicitly that they are independent of time.

4. Given a field $\phi(t, x)$ with a Lagrangian

$$L = \sum_n \left(\partial_t \phi(t, x_n) - v \frac{(\phi(t, x_{n+1} - \phi(t, x_{n-1})}{2\Delta x} \right)^2 - \left(\frac{(\phi(t, x_{n+1}) - \phi(t, x_{n-1})}{x_{n+1} - x_{n-1}} \right)^2 \quad (2)$$

where $x_0 = 0$, $x_N = L$, $\Delta x = x_n - x_{n-1}$ is independent of n, and $x_{n+N} \equiv x_n$. (periodic boundary conditions).

(This is the model for sound waves in a one dimensional lattice with unit sound velocity).

a) Find the conjugate momenta to $\phi(x_n)$ and the Hamiltonian for this problem.

b)Find the equations of motion for this field. If a mode for the field has the solution $\phi(t, x) \propto e^{-i(\omega t - kx)}$ what are the possible values of k and the relation between ω, k for a solution to the equations of motion.

c) What is the inner product for two of these solution-modes with different k and ω.

d) What is the norm of a mode for a given value of ω, k. What is the relation between the sign of the norm and that of ω if $v < \frac{2}{pi\Delta x}$, if $v < 1/2\Delta x$ and for $v > 1/2\Delta x$?

Note that this would be even more interesting if we were to make the dispersion relation look like that for a BEC

$$\begin{aligned} L &= \Delta x \sum_n \left(\partial_t \phi(t, x_n) - v \frac{(\phi(t, x_{n+1} - \phi(t, x_{n-1})}{2\Delta x} \right)^2 && (3) \\ &- \left(\frac{(\phi(t, x_{n+1}) - \phi(t, x_{n-1})}{x_{n+1} - x_{n-1}} \right)^2 && (4) \\ &- \kappa^2 \left(\frac{(\phi(t, x_{n+2}) - 2\phi(t, x_n) - \phi(t, x_{n-2})}{\Delta x^2} \right)^2 && (5) \end{aligned}$$

which is the discrete Gross Piatevskii approximation to a flowing Bose Einsten Condensate where $\kappa << N\Delta x$ is the healing length of the BEC. But this is not the question I ask.