Physics 501-20
Assignment 1

1.) Hardy system: Given the state

$$|\Psi\rangle = \alpha\,|\uparrow\rangle\,|1\rangle + \beta\,|\downarrow\rangle\,(S\,|1\rangle + C\,|0\rangle) \tag{1}$$

where this is a unit vector with $|\alpha|^2 + |\beta|^2 = 1$ and $|C|^2 + |S|^2 = 1$

i)Argue that we can always choose the coefficients as real and positive.

ii) Find the value of S that minimizes the probability of having the final value of "Y" be equal to +1. (Recall from the lectures that one has two systems, with A,B being attributes of the first system, and X,Y of the second. They are such that $A \to 1 \Rightarrow X \to 1, X \to 2 \Rightarrow B \to 1, B \to 2 \Rightarrow Y \to 1$, but $A \to 1$ does not imply that $Y \to 1$ (in fact the probability that when A has value 1 it is highly improbably that Y also has the value 1. $A \to 1$ here means A is found to have value 1. $\Rightarrow$ means "implies that"− ie if one makes measurements on the system, then it is always true that if A and X are measured, then whenever A is found to value 1, X always also has value 1.)

iii) Given that value of S, what is the largest value of the the ratio of the eigenvalues $\lambda_1,\ \lambda_2$ where the two λ are the two eigenvalues of the reduced density matrix of particle 1 with $\lambda 1$ being the smallest of the eigenvalues.

(recall that the density matrix for the second particle associated with a pure entangled state on the whole system is

$$|\Psi\rangle = \sum_i \lambda_i\,|\phi_i\rangle\,|\psi_i\rangle \tag{2}$$

is

$$\rho = \sum_{i,j} \lambda_i^* \lambda_j\,\rangle\phi_i|\,|\phi_j\rangle\,|\psi_j\rangle\,\rangle\psi_i| \tag{3}$$

where $|\phi\rangle$ is a state for the first particle/system, while $|\psi\rangle$ is a state for the second particle/system. For the Hardy system, use the two component vector to find the matrix representing the reduced density matrix for the second particle.

2) No Cloning:

Alice claims that she can duplicate a state, such that if the state is $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ and $|\phi\rangle = \alpha\,|\downarrow\rangle + \beta\,|\uparrow\rangle$ is the same state for the second system, then that she can start with a state $|\psi\rangle\,|\downarrow\rangle$ and create the state $|\psi\rangle\,|\phi\rangle$ without knowing what the coefficients α or β. Is she right and if not, why not, and if yes, how?

3) Bell states and Quantum Teleportation:

i) Show that the four bell states over two two-level systems$i$ and j is a complete orthonormal basis. Basis vectors for each individual system are indicated by $|\uparrow_i\rangle$ and $|\downarrow_i\rangle$ where i indicates which two level system is being referred to.) (Note that the state $|\uparrow_1,\uparrow_2\rangle$ say is equivalent to $|\uparrow_1\rangle|\uparrow_2\rangle$)

$$|B_{ij0}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_i\downarrow_j\rangle - |\downarrow_i\uparrow_j\rangle) \tag{4}$$

$$|B_{ij1}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_i\downarrow_j\rangle + |\downarrow_i\uparrow_j\rangle) \tag{5}$$

$$|B_{ij2}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_i\uparrow_j\rangle - |\downarrow_i\downarrow_j\rangle) \tag{6}$$

$$|B_{ij3}\rangle = \frac{1}{\sqrt{2}}(|\uparrow_i\uparrow_j\rangle + |\downarrow_i\downarrow_j\rangle) \tag{7}$$

$$\tag{8}$$

Now interpret $|\uparrow\rangle$ as the $\sigma_z = 1$ eigenstates for the Pauli matrix, and $|\downarrow\rangle$ as the $\sigma_z = -1$ eigenstate of that same z Pauli matrix. Define $Z_i \equiv \sigma_{zi}$ and similarly for Z_j and $Z_{ij} = Z_i Z_j$. The $|B\rangle_{ij0}$ and $|B\rangle_{ij1}$ are -1 eigenstates of the Z_{ij} operator and the other two are +1 eigenstates of that same operator. Also $|B\rangle_{ij0}$ and B_{ij2} are -1 eigenstates of the operator which exchanges $|\uparrow_i\rangle$ with $|\downarrow_i\rangle$ and $|\uparrow_j\rangle$ with $|\downarrow_j\rangle$ (the $X_{ij} = \sigma_{ix}\sigma_{jx}$ operator) and the other two are the +1 eigenstates of this operator. Define $Q_{ij} = \frac{1}{2}(Z_{ij} + 1) + (X_{ij} + 1)$. It has eigenvalues or $\{0, 1, 2, 3\}$ and the eigenvectors are B_{ijN}.

Now, introduce a third system in an unknown arbitrary state$\alpha|\downarrow_3\rangle + \beta|\uparrow_3\rangle$, and assume that the initial state of particles 1 and 2 is $|B_{120}\rangle$

i) One now measures the operator Q_{13} What is the state of of second particle after this measurement for the four possible eigenvalues? Show that the probabilities each of the possible outputs are all equal no matter what the α, β are. Ie, the outcome of the measurement of Q_{13} give no hints as to the values of α and β. Show that for each measured outcome there is a simple unitary transformation which converts the state of the second particle into $\alpha|\downarrow_2\rangle + \beta|\uparrow_2\rangle$. Ie, one has transformed the unknown state of the first particle 3 into the same state for particle 2. Note that this is true even if the second particle is over at Alpha Centauri and particles 1 and 3 are on earth.

Ie, a joint measurement on particles 1 and 3 which gives no information about that unknown state, and the classical transmission of the outcome of that measurement of that apparently useless measurement to someone who has system 2, allows someone at the location of 2 to make an exact copy of that unknown state of system 3, but at the expense of destroying the original state of system 3. Ie, this does not run afoul of the "No-cloning" theorem. There are never two copies of the initial state of system 3.