Physics 501-20

Assymptotic Adiabatic expansion

Let us start with the Hamiltonian

$$H = \frac{1}{2}(p^2 + \Omega^2 x^2) = \frac{\Omega}{2}(\frac{p^2}{\Omega} + \Omega x^2) \tag{1}$$

where Ω is a function of time. The action is

$$S = \int p\frac{dx}{dt} - H dt \tag{2}$$

Let us define a new time by

$$\tau = \int \Omega dt \tag{3}$$

The action then becomes

$$S = \int p\dot{x} - \frac{1}{2}(\frac{p^2}{\Omega} + \Omega x^2) \tag{4}$$

where $\dot{} = \frac{d}{d\tau}$. If we now define new momentum and configuration variables

$$\tilde{p} = \frac{p}{\sqrt{\Omega}}; \qquad \tilde{x} = \sqrt{\Omega}x \tag{5}$$

we have the action

$$S = \int \sqrt{\Omega}\tilde{p}\frac{d(\tilde{x}/\sqrt{\Omega})}{d\tau} - \frac{1}{2}(\tilde{p}^2 + \tilde{x}^2) \tag{6}$$

$$= \int \tilde{p}\frac{d\tilde{x}}{d\tau} - \frac{1}{2}(\tilde{p}^2 + \tilde{x}^2 + 2\frac{d\sqrt{\Omega}}{d\tau}\frac{\tilde{p}\tilde{x}}{\sqrt{\Omega}}) \tag{7}$$

Thus

$$\tilde{H} = \frac{1}{2}(\tilde{p}^2 + \tilde{x}^2 + \frac{2}{\sqrt{\Omega}}\frac{d\sqrt{\Omega}}{d\tau}\tilde{p}\tilde{x}) \tag{8}$$

Now define

$$\hat{p} = \tilde{p} + \frac{1}{\sqrt{\Omega}}\frac{d\sqrt{\Omega}}{d\tau}\tilde{x} \tag{9}$$

$$\hat{x} = \tilde{x} \tag{10}$$

which gives

$$S = \int \left((\hat{\pi} - \frac{1}{\sqrt{\Omega}}\frac{d\sqrt{\Omega}}{d\tau}\hat{x})\frac{d\hat{x}}{d\tau} - \frac{1}{2}\left(\hat{p}^2 + \hat{x}^2(1 - \left(\frac{\dot{\sqrt{\Omega}}}{\sqrt{\Omega}}\right)^2)\right)\right) d\tau \tag{11}$$

$$= \int \hat{p}\frac{d\hat{x}}{d\tau} - \frac{1}{2}(\frac{d(\frac{d\sqrt{\Omega}}{d\tau}/\sqrt{\Omega}\hat{x}^2)}{d\tau} + \frac{1}{2}(\hat{p}^2 + \hat{x}^2\left(1 - \frac{\ddot{\Omega}}{2\Omega} + \left(\frac{\dot{\Omega}}{4\Omega}\right)^2\right))d\tau \tag{12}$$

The second term is a complete derivative and thus picks up only the boundary terms in the action. Since the Hamiltonian action is defined so that the variations on the boundary are zero, this term contributes a term whose variation is zero to the action– ie, it does not alter the equations of motion. Thus we find that the Hamiltonian (with τ as the time) is

$$\hat{H} = \frac{1}{2}(\hat{p}^2 + \hat{x}^2(1 - \frac{\ddot{\sqrt{\Omega}}}{\sqrt{\Omega}}) \tag{13}$$

We note that we end up with the same form of the Hamiltonian as we began with now τ as the time and

$$\hat{\Omega}^2 = 1 - \frac{\ddot{\sqrt{\Omega}}}{\sqrt{\Omega}} = 1 - \frac{1}{\sqrt{\Omega}}\frac{d^2\sqrt{\Omega}}{d\tau^2} \tag{14}$$

If $1 - \frac{1}{\sqrt{\Omega}}\frac{d^2\sqrt{\Omega}}{d\tau^2} << \Omega^2$ this leads to getter approximation for the solution. This terms could be very small if Ω varies slowly with respect to time. Ie, this is an adiabatic approximation, if one neglects the difference between $\hat{\Omega}$ and 1.

One can run around this loop again with the ˆ variables, and continue this ad-infinitum. For the first few iterations, this will give a better and better approximation to the solution. However one can show that after a number of iterations the remaining effective Ω begins to diverge from 1, rather than converging. Ie, this sequence of solutions is an assymptotic expansion. The approximation scheme gets better and better for the first few steps, and then ,after a while, the approximation gets worse and worse.