Heisenberg Rep.

$$\dot{B} = -i[B, H]$$

H is quadratic operators.

$\hbar = 1$

$X, P \qquad [X,P] = i$

$$H = \frac{1}{2}(P^2 + \Omega^2 X^2) \quad (AP^2 + 2BPX + CX^2)$$

$\dot{X} = P \quad ; \quad \dot{P} = -\Omega^2 X$

$\dot{x} = p \quad ; \quad \dot{p} = -\Omega^2 x$

add solns another soln.

Soln to class eqn. real or complex.

q_1, p_1 ; q_2, p_2

$q_1 + i q_2,\ p_1 + i p_2$ is soln.

$q,\ q'$

$p,\ p'$

$$\langle \{q,p\}, \{q',p'\}\rangle = +\frac{i}{2}\left(q^* p' - p^* q'\right)$$

$$\langle \{q,p\}, \{q,p\}\rangle = +\frac{i}{2}\left(q^* p - p^* q\right)$$

Real

If q, p real then $\langle \{q,p\}, \{q,p\}\rangle = 0$

$$\langle \{q, p\}, \{q', p'\} \rangle \equiv \langle q, q' \rangle$$

Defn

$$\langle q, q \rangle = -\langle q^*, q^* \rangle$$

$$\langle q^*, q^* \rangle = \frac{i}{2}\left(q p^* - p q^* \right)$$

$$= -\langle q, q \rangle$$

$$\frac{d}{dt}\langle q, q' \rangle = 0$$

$$\frac{d}{dt}\left(q^* p' - p^* q'\right)$$

$$= \dot{q}^* p' + q^* \dot{p}' - \cdots$$

$$\frac{\partial H}{\partial p^*} p^* p' + q^* \frac{\partial H}{\partial q'}(p', q') - \cdots$$

$$= A p^* p' + B p^* q' + C q^* q'$$

$$- (\qquad) = 0$$

Norm is conserved.

Classical solutions are modes.

neg, pos norm modes

$$q, p \to \frac{q}{\sqrt{|\langle q, q\rangle|}}, \frac{p}{\sqrt{|\langle p, p\rangle|}}$$

unit norm.

unit norm modes. $\pm 1, 0$

P, Q $\quad \dot{P} = -i[P, H] \quad P^\dagger = P$

$\dot{Q} = -i[Q, H] \quad Q^\dagger = Q$

Classical Soln. $\{q, p\}$

$\langle q, q \rangle = +1$

$\frac{d}{dt} \langle q, Q \rangle = 0$; operator

$A = \langle q, Q \rangle = +\frac{i}{2}(q^* P - p^* Q)$

$[Q, P] = i$

$$[A, A^\dagger] = +\frac{1}{4}\left[(q^* P - p^* Q), \; qP - pQ)\right]$$

$$= \frac{i}{2}\left(q^* p - p^* q\right)$$

$$= \langle q, q \rangle = 1$$

$$(A^\dagger A)^\dagger = (A^\dagger A^{\dagger\dagger}) = A^\dagger A$$

$$\langle \psi | A^\dagger \underbrace{A | \psi \rangle}_{|\phi\rangle} = \langle \phi | \phi \rangle \geq 0$$

$$A^{+}A|r\rangle = r|r\rangle$$

$$A^{+}A(A|r\rangle) = (r-1)\,A|r\rangle$$

$$A^{s}|r\rangle = \underbrace{(r-s)}_{-ve.}\,A^{s}|r\rangle$$

if $r-s=0$ then $A^{s}|r\rangle = 0$

r must be +ve integer.

For <u>any</u> pos. norm mode

A_q, $\exists$ state $|0\rangle_q$

$$A_q |0\rangle_q = 0$$

In general, $A_q \neq A_{q'}$

$A_{q'}$ must a linear comb of A_q, $A_q^\dagger$

$$A_q' = \alpha A_q + \beta A_q^\dagger$$

$$[A_q', A_q'^\dagger] = [\alpha A_q + \beta A_q^\dagger, \alpha^* A_q^\dagger + \beta^* A_q]$$

$$= \alpha\alpha^* - \beta\beta^* = 1$$

$$|\alpha| = \cosh\theta, \quad |\beta| = \sinh\theta$$

$$A_q = \cosh\theta \, e^{i\phi} A_q + \sinh\theta \, e^{-i\phi} A^\dagger$$

$|0\rangle_{q'} \neq |0\rangle_q$

$A' = \cosh\theta\, A + \sinh\theta\, A^+$

$A|0\rangle = 0$

$A'|0\rangle' = 0$ in terms of $|0\rangle$

$|0\rangle' = f(A^+)|0\rangle$

$0 = A'|0\rangle' = (A\cosh\theta + A^+\sinh\theta) f(A^+)|0\rangle$

$$[A, A^{+n}]$$

$$= [A, A^{+}]A^{+n-1} + A[A, A^{+}]A^{n-2} + \cdots A^{+n-1}[A, A^{+}]$$

$$= n A^{+n-1}$$

$$A \rightarrow \frac{\partial}{\partial A^{+}}$$

$$\left(\frac{\partial}{\partial A^{+}} f(A^{+}) \cosh\theta + A^{+} f(A^{+}) \sinh\theta\right)$$

$$= 0$$

$$f(A^\dagger) = e^{-\tanh\theta \frac{A^{\dagger 2}}{2}}$$

$$\partial_y f + \tanh\theta \, y f = 0$$

$$f = e^{-\frac{\tanh\theta \, y^2}{2}}$$

Normalise it.

$$|0\rangle_{q'} = e^{-\frac{\tanh\theta \, A_q^{\dagger 2}}{2}} |0\rangle_q$$

$|0\rangle_{q'}$ is sum of double excited states of $|0\rangle_q$

$|0\rangle_{q'}$ is "squeezed" state of $|0\rangle_q$

$\frac{(A_{q'}+A^\dagger_{q'})}{\sqrt{2}}$, $i\frac{(A_{q'}-A^\dagger_{q'})}{\sqrt{2}}$ in $|0\rangle_{q'}$

Each +v norm mode has a "vacuum" state assoc with it $|0\rangle_{q'}$, and a many "particle" state assoc. with it.

Diff modes have diff. "vacuum" states, and diff "particle" states.

The transf.

$$A_q' = A_q \cosh\theta\, e^{i\phi} + \sinh\theta\, e^{i\psi} A_q^+$$

is called a Bogolubov transformation.

If at ~~some time~~

$$\left.\begin{aligned} \dot{q} &= -i\omega q \\ \dot{p} &= -i\omega p \end{aligned}\right\} \text{mode for some } \omega.$$

$$\langle q, q \rangle = \omega |q|^2 \qquad \omega > 0.$$

Hamiltonian diagonalisation mode

$$|0\rangle_q = |\text{min energy state}\rangle$$

$$\left(H(t)|0\rangle = E|0\rangle \right)$$

There could exist different min energy states from time to time.

"Vacuum state"

Generalize to multi systems.

q_i, P_i