Physics 407-07
Green's Function solution to wave equation

We want to find the Green's function solution to the equation

$$\Box\Psi(t,\mathbf{x};t',\mathbf{x}') = \delta(t-t')\delta^3(\mathbf{x}-\mathbf{x}') \tag{1}$$

where $\mathbf{x}$ represents the three coordinates x, y, z.

We can take the Fourier transform of both sides

$$\Psi(\omega,\mathbf{k}) = \int \Psi(t,\mathbf{x};0,0)e^{i(\omega t-\mathbf{k}\cdot\mathbf{x})}dtd^3x \tag{2}$$

where $\mathbf{k}$ represents k_x, k_y, k_z, to get

$$(-\omega^2 + \mathbf{k}\cdot\mathbf{k})\Psi(\omega,\mathbf{k}) = 1 \tag{3}$$

or

$$\Psi(\omega,\mathbf{k}) = \frac{-1}{\omega^2 - \mathbf{k}\cdot\mathbf{k}} \tag{4}$$

The

$$\Psi(t,\mathbf{x};t'\mathbf{x}') = \int\int\int\int \frac{-1}{\omega^2-\mathbf{k}\cdot\mathbf{k}}e^{-i(\omega(t-t')-\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\frac{1}{(2\pi)^4}d\omega d^3\mathbf{k} \tag{5}$$

Writing $\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') = k|\mathbf{x}-\mathbf{x}'|\cos(\theta)$, where $k^2 = k_x^2 + k_y^2 + k_z^2$, θ is the angle between the $\mathbf{k}$ vector and the $\mathbf{x}$ vector, ϕ is the additional azimuthal angle, and $d^3\mathbf{k} = k^2dkd(\cos(\theta))d\phi$ we have

$$\Psi(t,\mathbf{x};t'\mathbf{x}') = \frac{-1}{(2\pi)^3}\int\int\int \frac{e^{-i(\omega(t-t')+k|\mathbf{x}-\mathbf{x}'|cos(\theta))}}{\omega^2-k^2}k^2dkd(\cos(\theta))d\omega \tag{6}$$

$$= \frac{-1}{(2\pi)^3}\int\int \frac{1}{\omega^2-k^2}e^{-i\omega(t-t')}\frac{e^{ik(\mathbf{x}-\mathbf{x}'|)} - e^{-ik(\mathbf{x}-\mathbf{x}'|)}}{ik|x-x'|}k^2dkd\omega \tag{7}$$

$$= \frac{-1}{i|\mathbf{x}-\mathbf{x}'|(2\pi)^3}\int\int \frac{1}{\omega^2-k^2}e^{-i\omega(t-t')}(e^{ik|\mathbf{x}-\mathbf{x}'|} - e^{-ik|\mathbf{x}-\mathbf{x}'|})kdkd\omega \tag{8}$$

We now do the integral over ω by doing a contour integral. We want the contour at infinity to contribute nothing, so $e^{i\omega(t-t')}$ must go to zero as ω assumes imaginary parts at infinity. This means that we must close the contour to positive imaginary ω for $t-t'>0$ and to negative imaginary parts for $t-t'<0$. We also want the integral to be zero for $t<t'$, so that the influence of the source is to the future, not the past. This means we must take the contour such that if we enclose it to negative imaginary ω, it must not enclose any of the singularities, which means that we need to take the contour along the real axis so that it runs below both of the singularities.

Thus

$$\Psi(t,\mathbf{x};t';\mathbf{x}') = \frac{-1}{i|\mathbf{x}-\mathbf{x}'|(2\pi)^3}\int_0^\infty (2\pi i)\left(\frac{e^{-ik(t-t')}}{2k}-\frac{e^{ik(t-t')}}{2k}\right)\left(e^{ik|\mathbf{x}-\mathbf{x}'|}-e^{-ik|\mathbf{x}-\mathbf{x}'|}\right)kdk\Theta(t-t') \tag{9}$$

$$= \frac{-1}{|\mathbf{x}-\mathbf{x}'|2(2\pi)^2}\int_0^\infty (-e^{ik(t-t'-|\mathbf{x}-\mathbf{x}'|)}+e^{-ik(t-t'-|\mathbf{x}-\mathbf{x}'|)})+(e^{ik(t-t'+|\mathbf{x}-\mathbf{x}'|)}-e^{-ik(t-t'+|\mathbf{x}-\mathbf{x}'|)})dk \tag{10}$$

$$= \frac{\Theta(t-t')}{|\mathbf{x}-\mathbf{x}'|2(2\pi)^2}\int_{-\infty}^\infty e^{ik(t-t'-|\mathbf{x}-\mathbf{x}'|)}-e^{ik(t-t'+|\mathbf{x}-\mathbf{x}'|)}dk \tag{11}$$

$$= \frac{\Theta(t-t')}{4\pi|\mathbf{x}-\mathbf{x}'|}\left(\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)-\delta(t-t'+|\mathbf{x}-\mathbf{x}'|)\right) \tag{12}$$

where $\Theta(t-t')=1$ if $t-t'>0$ and is zero if $t-t'<0$

But, since both $t-t'$ and $|x-x'|$ are positive, the second delta function can never have zero argument. This means that it is always zero. Thus

$$\Psi(t,\mathbf{x};t';\mathbf{x}') = \frac{\Theta(t-t')}{4\pi|\mathbf{x}-\mathbf{x}'|}\delta(t-t'-|\mathbf{x}-\mathbf{x}'|) \tag{13}$$

as required.

Note that there is a more invariant way of writing this

$$\Psi(t,\mathbf{x};t';\mathbf{x}') = -\frac{\Theta(t-t')}{2\pi}\delta((t-t')^2-|x-x'|^2) \tag{14}$$

since

$$\int f(x)\delta(g(x))dx = \int f(x(g))\delta(g)\frac{dg}{\frac{dg(x)}{dx}} = \sum_i \frac{f(x_i)}{\frac{dg(x_i)}{dx}} \tag{15}$$

where $g(x_i)=0$. Thus

$$\int \frac{\Theta(t-t')}{2\pi}\delta((t-t')^2-|x-x'|^2)F(t')dt' \tag{16}$$

$$= \frac{1}{4\pi}\left(\frac{\Theta(|x-x'|)F(t-|x-x'|)}{|x-x'|}+\frac{\Theta(-|x-x'|)F(t+|x-x'|)}{-|x-x'|}\right) \tag{17}$$

since $\partial_{t'}((t-t')^2 - |x-x'|^2) = -2(t-t')$ and evaluated at the two zeros of $((t-t')^2 - |x-x'|^2)$ for t' this is $-2(\pm|x-x'|)$

This latter form of the Green's function is far more clearly a Lorentz invariant form. The argument of the delta function is clearly Lorentz invariant, and the $\Theta(t-t')$ simply picks out the positive null cone rather than the negative, which is also clearly Lorentz invariant.