Physics 407-07
Assignment 2

1.) Consider the two dimensional flat metric

$$ds^2 = dx^2 - dt^2 \tag{1}$$

and the coordiante transformation

$$x = \rho\cosh(\tau) \tag{2}$$
$$t = \rho\sinh(\tau) \tag{3}$$

Show that the curve $\rho = const$ is a timelike curve and $\tau = const$ a spacelike one.

Now take the equations

$$x = T\sinh(Y) \tag{4}$$
$$t = T\cosh(Y) \tag{5}$$

Which region of flat space-time in $x,\ t$ do these coordinates cover? What are the equations of straight lines for these equations?

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Consider $\rho = \rho_0 = const$ so the equation of the curve parameterized by τ is

$$x = \rho_0\cosh(\tau) \tag{6}$$
$$y = \rho_0\sinh(\tau) \tag{7}$$

so the length squared of a little piece of the curve is

$$ds^2 = -(\partial_\tau t)^2 + (\partial_\tau x)^2 = -\rho_0^2 < 0 \tag{8}$$

so it is spacelike.

Similarly if we choose a curve with $\tau = \tau_0 = const$ and use ρ as the parameter, we have

$$ds^2 = -(\partial_\rho t^2 + (\partial_\rho x)^2 = 1 > 0 \tag{9}$$

The equations of $T\ Y$ coordinates can be written as

$$x^2 - t^2 = T^2(sinh^2(Y) - cosh^2(Y)) = -T^2 \tag{10}$$

Ie, the displacement of any point in the $T\ Y$ space from the origin is timelike. Ie, these points cover the future and past light cone of the origin of the xy coordinates.

——————————————————————————- There are a very large number of ways of solving this problem of straight lines.

a)The easiest way to find the straight lines is to go to t x coordinates. We know what the solutions for straight lines here is

$$x - vt = x_0$$

or

$$T\sinh(Y) - Tv\cosh(T) = x_0$$

If $|v| < 1$ then defining $\sinh(Y_0) = v/\sqrt{1-v^2}$, we have

$$T(\cosh(Y_0)\sinh(Y) - \sin(Y_0)\cosh(Y)) = \sqrt{1-v^2}x_0$$

or

$$T\sinh(Y - Y_0) = \frac{x_0}{\sqrt{1-v^2}}$$

If $|v| > 1$, then we can set $\frac{cosh(Y_0)=v}{\sqrt{v^2-1}}$ and

$$T\cosh(Y - Y_0) = \frac{x_0}{\sqrt{v^2-1}}$$

Finally, if $|v| = 1$, then

$$Te^{\pm Y} = x_0$$

If we want the dependence on s, we know that

$$t = \alpha(s - s_0)x = x_0 + v\alpha(s - s_0)$$

and since $(\partial_s t)^2 - (\partial_s x)^2 = \alpha^2(1 - v^2)$, we have $\alpha^2 = \frac{1}{|1-v^2|^2}$ (if $v^2 \neq 1$) and we can substitute these into the above equations to get T and Y in terms of s.

Note that some may worry that this is "unfair". It is not. One of the key features of General Relativity is that if you can find a coordinate system in which the problem is easy to solve, solve it in that coordinate system and then transform to the coordinate system you need the solution in.

b) Solve the geodesic (straight line) equations in the coordinate system. The straight line equations can be obtained from the metric

$$\begin{aligned} ds^2 &= dx^2 - dt^2 = (d(T\sinh(Y))^2 - (d(T\cosh(Y))^2) && (11)\\ &= -dT^2 + T^2 dY^2 && (12) \end{aligned}$$

$$\begin{aligned} \delta I &= \delta \int -((\partial_s T)^2 + T^2(\partial_s Y)^2 ds && (13)\\ &= \int 2(\partial_s^2 T + 2T(\partial_s Y)^2 \delta T - 2\partial_s(T\partial_s Y)\delta Y && (14) \end{aligned}$$

from which

$$\begin{aligned} &\partial_s^2 T + 2T(\partial_s Y)^2 = 0 && (15)\\ &partial_s(T^2\partial_s Y) = 0 && (16) \end{aligned}$$

The second is easily solved,

$$\partial_s Y = \frac{C}{T^2} \tag{17}$$

where C is a constant. Substituted into the first gives

$$\partial_s^2 T + \frac{C^2}{T^3} = 0 \tag{18}$$

Multiplying by $\partial_s T$ this is a complete derivative which has as first integral

$$(\partial_s T)^2 - \frac{C^2}{T^2} = D \tag{19}$$

wehre D is a constant. Defining s as the path length, this will be either ± 1 or zero depending on whether the curve is a spacelike, timelike or null curve.

The solution is

$$\int \frac{TdT}{\sqrt{DT^2 + C^2}} = s - s_0 \tag{20}$$

$$T = \sqrt{D(s - s_0)^2 + C^2} \qquad , D = \pm 1 \tag{21}$$

$$T = \sqrt{2|C|(s - s_0)} \qquad , D = 0 \tag{22}$$

The solution for Y is then

$$Y - Y_0 = \int \frac{C}{T^2} ds \tag{23}$$

$$= \frac{1}{2} ln(s - s_0) = ln(\frac{T}{\sqrt{2C}} \qquad D = 0 \tag{24}$$

$$Y - Y_0 = C tan^{-1}(\frac{s - s_0}{C}) \qquad D = 1 \tag{25}$$

$$Y - Y_0 = 2C \ln(\frac{s - s_0 - C}{s - s_0 + C}) \qquad D = -1 \tag{26}$$

Alternatively we rewrite the equation for T with Y as the parameter instead of s.

$$\partial_x T = \partial_Y T \partial_s T = \partial_Y T \frac{C}{T^2}$$

so

$$\frac{((\partial_Y T)^2 - T^2)(C}{T^2)^2 = D}$$

or

$$\int \frac{dT}{\sqrt{\frac{DT^4}{C^2} + T^2}} = Y - Y_0$$

While this looks hard to integrate, set $\tau = 1/T$ to get

$$\int \frac{d\tau}{\sqrt{\frac{D}{C^2 + \tau^2}}} = Y - Y_0$$

This gives the same solutions as before.

** 2. Calculate the Newtonian potential gravitational potential for an infinite plane of matter. (assume that the solution of Newton's equations is independent of y and z) What would Einstein's guess at a metric for this distribution of matter be. Compare this to the coordinates for Rindler space given above. For the Newtonian case, does the surface $\rho = 0$ from problem 1 make any sense?

———————————————————————— for an infinite plane of matter,

$$\nabla^2\phi = -4\pi G\sigma_0\delta x$$

where σ is the surface density of mass in the plane.

$$\phi = -2\pi G\sigma_0(|x|)$$

Einstein's initial guess would have made the temporal part of the metric be $(1-\phi)^2$ (with c=1)

$$ds^2 = -(1+2\pi G\sigma_0|x|)^2 dt^2 + (dx^2 + dy^2 + dz^2) \tag{27}$$

Note that this is exactly the metric for flat spacetime in the accelerated coordinates above on each side of $x = 0$. In fact this is the form of the exact solution of Einstein's equations outside a flat plane of matter. It is just flat spacetime in the accelerated coordinates

(Of course to lowest order in the potential, we would write this as

$$ds^2 = -(1+4\pi G\sigma_0|x|)dt^2 + (dx^2 + dy^2 + dz^2) \tag{28}$$

for small x.)

3. Consider 3 dimensional flat space in rotating coordinates

$$ds^2 = dx^2 + dy^2 - dt^2 \tag{29}$$

First define polar spatial coordinates

$$x = r\cos(\theta) \tag{30}$$

$$y = r\sin(\theta) \tag{31}$$

What is the metric in this coordinate system?

Now define a new ϕ coordinate by

$$\theta = \phi - \omega t \tag{32}$$

What is the metric in this coordinate system?

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$$ds^2 = -dt^2+dx^2+dy^2 = -dt^2+dr^2+r^2d\theta^2 = -dt^2+dr^2+r^2(d\phi-\omega dt)^2 = -dt^2(1-\omega^2r^2)-2r^2\omega d\phi dt+dr^2+r^2d\phi^2$$

The gravitational potential would thus be (assuming $\omega r << 1$ which is the only case here that Newtonian treatement would makde sense)

$$\Phi = \frac{1}{2}\omega^2 r^2 \tag{33}$$

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A particle at constant ϕ, r goes around in circles in the x, y coordinates. What is the gravitational potential for this particle? What are the geodesic equations for a particle in this system of coordinates?

Consider the particle travelling along a geodesic. For small velocities of the particle, there is an apparent force on the particle, which depends both on position and velocity. What is that force?

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The geodesic equations are

$$\partial_s((1-\omega^2 r^2)\partial_s t + \omega r^2 \partial_s \phi) = 0 \tag{34}$$
$$\partial_s^2 r - \omega^2 r(\partial_s t)^2 + 2\omega r \partial_s t \partial_s \phi - r(\partial_s \phi)^2 = 0 \tag{35}$$
$$\partial_s(r^2 \partial_s \phi - \omega r^2 \partial_s t) = 0 \tag{36}$$

Trying to determine the forces is actually much harder than I thought it was.

The first and third equations can be integrated easily to give

$$(1-\omega^2 r^2)\partial_s t + \omega r^2 \partial_s \phi = p_t \tag{37}$$
$$r^2 \partial_s \phi - \omega r^2 \partial_s t = p_\phi \tag{38}$$

from which we get

$$\partial_s t = p_t - \omega p_\phi \tag{39}$$
$$\partial_s \phi = \frac{p_\phi}{r^2} + \omega(p_t - \omega p_\phi) \tag{40}$$

from which we get, keeping only terms to lowest order in $\partial_s \phi$ and $\partial_s r$

$$\frac{\partial_s(r\partial_s(\phi)) = -(p_\phi}{r^2)\partial_s r} \tag{41}$$

Now to zeroth order in the velocities, $\frac{p_\phi}{r^2} = \partial_s t = \frac{1}{\sqrt{1-\omega^2 r^2}}$ so

$$\partial_s(r\partial_s(\phi)) = \frac{1}{\sqrt{1-\omega^2 r^2}}\partial_s r \tag{42}$$

The apparent force on the angular velocity is proportional to the velocity in the radial direction. Similarly,

$$\partial_s^2 r = \omega^2 r(\partial_s t)^2 + 2\omega r \partial_s t \partial_s \phi + r(\partial_s \phi)^2 \tag{43}$$
$$= r(\partial_s \phi - \omega \partial_s t)^2 = r(\frac{p_\phi}{r^2})^2 \tag{44}$$

This one is trickier since we have to keep terms to first order in the velocities on the right side. From the definition equation for the length, we find that

$$\partial_s t \approx \frac{1}{\sqrt{1-\omega^2 r^2}} - \omega r^2 \frac{\partial_s \phi}{1-\omega^2 r^2} \tag{45}$$

and

$$\partial_s^2 r = r(\frac{\omega^2}{sqrt1-\omega^2 r^2}) + \frac{\omega r}{1-\omega^2 r^2}\partial_s \phi \tag{46}$$

The radial force has a term which is independent of the velocity, and another which is proportional to the velocity in the ϕ direction. The velocity dependent terms are just the Coriolis forces, while the other is the "centripetal" force.

(By "force" I mean the right hand side of the equations–the rate of change of physical velocity as a function of time.)

$$\frac{d^2 r}{dt^2} = \ldots \tag{47}$$

$$\frac{d}{dt}\left(r\frac{d\theta}{dt}\right) = \ldots \tag{48}$$

4.) Equivalence Principle:

Show that the Eotvos experiment still works even if the two masses are not the same. Ie, assume you have two objects with gravitational mass m_1 and m_2 and inertial masses of $(1+\epsilon_1)m_1$ and $(1+\epsilon_2)m_2$ hung as a torsion balance of length L from a fibre so that if they are oriented north-south, they hang horizontally and are oriented exactly North-south. If the support is rotated exactly 90 degrees, find the angle with the east-west direction that the arm of the balance hangs at, and show that this angle is proportional to $\epsilon_1 - \epsilon_2$. What is the deflection angle? (Assume that the period of torsional oscillation of the system is T seconds and that the laboratory is located at latitude θ.) You can assume that ϵ is very small and keep only first order effects in ϵ.

The key point is that the torsion bar, hanging horizontally must have the lengths of the arms distributed so that the horizontal torque is is zero. Thus to lowest approximation, $l_1 m_1 g = l_2 m_2 g$. The vertical torque is what causes the deflection. The centripital "force" torque is

$$\mathcal{T} = (l_1(1+\epsilon_1)m_1\Omega^2 Rsin(\theta)cos(\theta) - l_2(1+\epsilon_2)m_2\Omega^2 Rsin(\theta)cos(\theta))sin(\phi) \tag{49}$$

where R=radius of the earth, θ is the latitude, ϕ is the angle from the NS direction of the orientation of the balance. Now since $l_1 m_1 = l_2 m_2$ we have

$$\mathcal{T} = \frac{1}{2} l_1 m_1 (\epsilon_1 - \epsilon_2)\Omega^2 Rsin(2\theta)sin(\phi) \tag{50}$$

If $l = l_1 + l_2$, we have $l_1 m_1 = \frac{m_2}{M} l$ where $M = m_1 + m_2$. and the torque is

$$\mathcal{T} = \frac{m_1 m_2}{M} l(\epsilon_1 - \epsilon_2)\Omega^2 Rsin(2\theta)sin(\phi) \tag{51}$$

Note that there are a number of possible corrections. Because of the centripital force, the vertical is not quite the true vertical, but is corrected by that force. Since it includes the effect of the gravitiational/inertial mass imbalance, this result in a small correction. It would however have only a quadratic correction on $\epsilon_1 - \epsilon_2$ which is unmeasureable. When the balance lies NS, the centripital accelearation on the north most mass is slightly less than on the southmost because if the bar is horizontal, the north most mass is slightly nearer the center of the earth than the southmost. However this exerts a torque on the pendulum which lies in the horizontal plane, and thus has no effect on the deflection.

Now when that vertical torque is present, it must be offset by a deflection of the fibre. If the Torque constant is k, we must have

$$\mathcal{T} = k\delta\psi \tag{52}$$

or

$$\delta\psi = \frac{\epsilon_1 - \epsilon_2)}{2k}\frac{m_1 m_2}{M} l\Omega^2 R sin(2\theta) sin(\phi) \tag{53}$$

To estimate k we note that the equation of motion of the torsion pendulum with torsion constant k and moment of intertia $I = m_1 l_1^2 + m_2 l_2^2 = \frac{m_1 m_2}{M} l^2$ is $\omega = \sqrt{\frac{k}{I}}$ or $k = (\frac{T}{2\pi})^2 I$

If the length of the torsion bar is 1 meter, m_1 is approximately m_2, and approximately 1Kg, and the torsional period is 40 min, what is the value of the deflection angle as a function of $\epsilon_1 - \epsilon_2$. Assuming that the graduate student sent into the room to measure the angles has a mass of 100Kg and is 10m from m_1 perpendicular to the torsion arm, what would be the deflection caused by the student in comparison with the deflection caused the difference in inertial/gravitational mass, if $\epsilon_1 - \epsilon_2$ is 10^{-8}. .

From the above (and assuming that the experiment takes place at lattitude 45^o) we have

$$\mathcal{T} = (\epsilon_1 - \epsilon_2)(.5kg)(1m)(6 \cdot 10^6 m)(\frac{2\pi}{86400sec})^2 = (\epsilon_1 - \epsilon_2)1.6 \cdot 10^{-2} \approx 2 \cdot 10^{-10} \tag{54}$$

The effect of the graduate student would be:

The force on the mass he would be closest to would be $G m_1 M_g / d_1^2$ while the force on the second mass would be $G m_2 M_g / d_2^2$ but at a slight angle to the perpendicular. The vertical torque would be

$$\mathcal{T}_g = G m_1 M_g / d_1^2 l_1 - cos(\mu) G m_2 M_g / (d_2)^2 = G M_g \frac{m_1 m_2}{M} l (1/d_1^2 - \frac{d_1}{(d_1^2 + l^2)^{3/2}}) \tag{55}$$

$$\approx G M_g \frac{m_1 m_2}{M}\frac{3l^3}{2d_1^4} \approx 6.7 \cdot 10^{-11}(50Kg)(.5kg)(1.5)(\frac{1m)^3}{10m)^4} = 2.5 \cdot 10^-13 \tag{56}$$

Ie, for these values, the effect of the graduate student is still significantly smaller than the effect being measured (If one is trying to get to $\epsilon_1 - \epsilon_2 = 10^{-12}$ as

Braginsky claimed, the effect of the graduate student is larger than the effect being measured Automated techniques which do not introduce large masses near the torsion pendulum are needed. Note that large 10 ton trucks driving up near the pendulum say 10m away could cause problems).