Physics 407-07
Assignment 1

1.) Given the metric

$$ds^2 = (x^2 dx^2 + x^4 dy^2) \tag{1}$$

find the equations for straight lines in this space.

Show that the lines $y = \mathit{constant}$ are solutions and find what x is as a function of s for these lines.

Change the x cordinate, so that for these solutions, the new coordinate would be a linear function of s. What would the metric be for this choice of x coordinate?

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Using the notion δ for ∂_ϵ, we can write the ϵ derivative, and using the action without the square root that we argues we can use if we choose λ to equal s along the solution curve, we have

$$\delta I = \int_{\lambda_1}^{\lambda_2} 2\delta x x(\partial_s x^2 + 2x^2 \partial_s y^2) + x^2(\partial_\lambda \delta x \partial_s x + \partial_\lambda \delta y x^4 \partial_s y) d\lambda \tag{2}$$

where I have used s for those places where we are on the solutions curve and λ where we have to remind ourselves that we are perhaps off the solution curve.

Taking the integration with respect to λ and recalling that the end point terms are zero ($\delta x = \delta y = 0$ at the endpoints) we have

$$\delta I = 2\int_{s_1}^{s_2} \qquad \delta x \left(x((\partial_s x)^2 + 2x^2(\partial_s y)^2) - \partial_s(x^2 \partial_s x)\right) \tag{3}$$

$$\delta y(-\partial_s(x^4 \partial_s y))ds \tag{4}$$

Now, if δI is supposed to be zero for all possible variations from the curve, then we must have

$$x((\partial_s x)^2 + 2x^2(\partial_s y)^2) - \partial_s(x^2 \partial_s x) = 0 \tag{5}$$

$$-\partial_s(x^4 \partial_s y) = 0 \tag{6}$$

Now, if $y = \mathit{const}$ all derivatives of y are zero, and the equations become

$$x((\partial_s x)^2) - \partial_s(x^2 \partial_s x) = 0 \tag{7}$$

$$0 = 0 \tag{8}$$

Again we know we have a first integral of

$$x^2(\partial_s x)^2 + x^4(\partial_s y)^2 = 1 \tag{9}$$

or

$$x^2(\partial_s x)^1 = 1 \tag{10}$$

or

$$\frac{1}{2}\partial_s(x^2) = \pm 1 \tag{11}$$

Thus, the distance along these curves is $x^2/2$. If we define $R = x^2/2$, the original metric becomes

$$ds^2 = dR^2 + 4R^2 dy^2 \tag{12}$$

If we interpret $2y$ as θ and R as r, this is exactly the metric of flat space in polar coordinates.

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2.) For flat spacetime, where in $x,\ y$ coordinates, the metric is

$$ds^2 = dx^2 + dy^2 \tag{13}$$

define new coordinates $X,\ Y$ such that

$$x = \cosh(X)\cos(Y) \tag{14}$$
$$y = \sinh(X) sin(Y) \tag{15}$$

Find the metric in terms of X and Y (I.e., express the distance ds^2 in terms of $dX,\ dY$ and functions of X and Y). (These coordinates are called elliptical coordinates because surfaces of constant X are ellipses, and of constant Y are hyperbolas)

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$$dx = \sinh(X)\cos(Y)dX - \cosh(X) sin(Y)dY \tag{16}$$
$$dy = \cosh(X)\sin(Y)dX + \sinh(X)\cos(Y)dY \tag{17}$$

Substituting this into the expression for the metric, we have

$$\begin{aligned} dx^2 + dy^2 &= (\sinh(X)\cos(Y)dX - \cosh(X) sin(Y)dY)^2 + (\cosh(X)\sin(Y)dX + \sinh(X)\cos(Y)dY)^2 && (18) \\ &= (\sinh^2(X)\cos^2(Y) + \cosh^2(X)\sin^2(Y))(dX^2 + dY^2) && (19) \end{aligned}$$

The question is whether or not any of the lines $X = const$ or Y=const. are straight lines.

Of course, the easy way is simply to look at what curves these represent in x,y coordinates. If we take $X = const$ we get $x/y = Const\ cos(Y)/sin(Y)$ which is a stright line only is that Const is zero or infinity. It cannot be zero, and so the only solution is X=0, which is the x axis.

Similarly if Y is constant, then $x/y = \cosh(X)/\sinh(X)\ Const$ which is a constant only if $Const$ is 0 or ∞. Again this is either the X or the Y axes.

Note that in both cases, the line os oly part of the X axis. (in the first case, lying between $-1 < x < 1$ and in the second $|x| > 1$.

One can also derive the equations of the straight line for this metric.

$$\partial_s(\sinh^2(X)\cos^2(Y) + \cosh^2(X)\sin^2(Y))\partial_s X) - (\cosh(X)\sinh(X)((\partial_s X)^2 + (\partial_s Y)^2) = 0 \quad (20)$$
$$\partial_s(\sinh^2(X)\cos^2(Y) + \cosh^2(X)\sin^2(Y))\partial_s Y) - (cos(Y)\sin(Y))((\partial_s X)^2 + (\partial_s Y)^2) = 0 \quad (21)$$

If we take $X = A$, then these equations become

$$\cosh(X)\sinh(X)(\partial_s Y)^2 = 0 \quad (22)$$
$$\partial_s(\sinh^2(X)\cos^2(Y) + \cosh^2(X)\sin^2(Y))\partial_s Y) - cos(Y)\sin(Y)(\partial_s Y)^2 = 0 \quad (23)$$

from the first, either $\cosh(X)\sinh(X) = 0$ which means that $X = 0$ or $\partial_s(Y) = 0$, which would mean that both and that is not a line at all– it is a point. Thus we must have that $X = 0$, which give the second equation as

$$\partial_s(\sin^2(Y)\partial_s Y) - cos(Y)sin(Y)(\partial_s Y)^2 = 0 \quad (24)$$

which certainly has a solution for Y as a function of s. iIe, $X = 0$ is a straight line.

Similarly if we take $Y = const$, the only solution is $Y = 0$.

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Are any of the lines $X = const$ or $Y = const$ straight lines? (You do not have to solve the equations but just show that they are straight lines)

3. Consider the metric

$$ds^2 = x^2dx^2 + x^2dy^2 \quad (25)$$

Find the geodesic equations for this metric.

Show that

$$x = iy + C \quad (26)$$

formally (for x and y complex) are solutions to these equations.

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The equations are

$$\partial_s(x^2\partial_x x) - x((\partial_x x)^2 + (\partial_s y)^2) = 0 \quad (27)$$
$$\partial_s(x^2\partial_s y) = 0 \quad (28)$$

If $x = iy$, substituting into these two equations gives

$$-i\partial_s(y^2\partial_s y) - iy(-(\partial_s y)^2 + (\partial_s y)^2) = 0 \quad (29)$$
$$-\partial_s(y^2\partial_s y) = 0 \quad (30)$$

But these are idential equations. Thus if we solve for y as a function os s it will satisfy both equations. Ie, it will be a consistant solution.

But this means that that is a "straight line" but of course not in real space. Note that these do NOT obey

$$x^2((\partial_x x)^2 + (\partial_s y)^2) = 1 \tag{31}$$

but rather that is equal to zero. Ie, this is like a null curver.

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4.a) A particle of mass M at rest emits a gamma ray of energy E, leaving a particle of mass M'. What is M' as a function of E and M?

Many ways of solving this.

$$\bar{P}_M = \bar{P}_\gamma + \bar{P}_{M'} \tag{32}$$

where P_M has only the t component non-zero, and it is Mc. Then

$$-M'^2c^2 = \bar{P}_{M'} \cdot P_{M'} = (\bar{P}_M - P_\gamma) \cdot (\bar{P}_M - \bar{P}_\gamma) \tag{33}$$

$$= \bar{P}.\bar{P}_M + \bar{P}_\gamma \cdot \bar{P}_\gamma - 2\bar{P}_M \cdot \bar{P}_\gamma \tag{34}$$

$$= -M^2c^2 - 0 + 2McE/c \tag{35}$$

or

$$M'^2 = M(Mc^2 - 2E)/c^2 \tag{36}$$

You could also solve by writing down the equations for the components. Assume that the gamma goes in the x direction. Then $P_{\gamma x} = E/c$ and

$$Mc = E_{M'}/c + E/c \tag{37}$$

$$0 = P_{M'x} + E/c \tag{38}$$

$$\tag{39}$$

with

$$-(E_{M'}/c)^2 + (P_{M'x})^2 = -(M'c)^2 \tag{40}$$

or

$$(Mc - E/c)^2 - (-E/c)^2 = M'^2c^2 \tag{41}$$

$$\tag{42}$$

with the same as before.

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b)Now the resultant particle of mass M' in part a) at rest. What energy ϵ would a gamma ray particle need in order to be absorbed and create the original particle of mass of M again. Why is ϵ not the same as E?

(I will use the same symbols as in a) but they do not mean the same thing)

$$\bar{P}_M = \bar{P}_\gamma + \bar{P}_{M'} \tag{43}$$

where now $P_{M't} = M'c$ is the only non-zero component and $\bar{P}_{\gamma t} = \bar{P}_{\gamma x} = \epsilon/c$.

In this case we want ϵ so we just square both sides and get

$$-M^2c^2 = -M'^2c^2 - 2\epsilon M' \tag{44}$$

or

$$\epsilon = (M^2 - M'^2)c^2/2M' = \frac{M}{M'}E \tag{45}$$

The extra energy is needed because of the recoil energies of the particles in the two cases. In the first case the mass difference goes into recoil energy and gamma energy. In the second the mass difference plus the recoil enegy is the gamma energy.

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5.) a) Pole and Barn revisited. A runner carrying a horizontal pole is running at a barn at 4/5 the velocity of light. When both are at rest, the pole has exactly the same length as the barn. According to the runner, the barn will be contracted. However, when the runner, located at the trailing edge of the pole, **sees** the front of his pole hit the far inside end of the barn, how far from the end of the barn is he located?

This is different from the usual in that what is important is what the runner sees, not what he infers. Thus, although the barn is shorter than the pole in his frame, it takes time for the light to get from the front of the pole hitting the back of the barn to the runner. If the length of the pole is L, it takes light L/c time to get to the runner. During that time the front of the barn (which was at $\sqrt{1-v^2/c^2}L$ in the runners frame when the front of th epole hit the back of the barn) moves vL/c. Thus the location of the front of the barn is $\sqrt{1-v^2/c^2}L + vL/c$ while the back of the barn is at vL/c.

Now $\sqrt{1-v^2/c^2} + v/c$ is always bigger than one, while v/c is less than one. Ie the runner sees the front of the pole hit the back of the barn when the runner is inside the barn.

If v/c=4/5. then $\sqrt{1-v^2/c^2} + v/c$ is 7/5L and v/cL is 4/5 L. The runner is located at L, so he will be 1/3 of the way from the back of the barn toward the front of the barn.

One can also do this same calculation from the barn's point of view. Here, the light travels fromt he back of the barn toward the runner at c, at the same time as the runnner is going toward the back at v. The runner is at the point $\sqrt{1-v^2/c^2}L$ when the pole hits the back. Thus the equation for the runner is (setting t=0 as the time when the pole hits the back) $\sqrt{1-v^2/c^2}L - vt = ct$, and thus the point where they meet is $ct = c\sqrt{1-v^2/c^2}L/(c+v) = \sqrt{\frac{c-v}{c+v}}L = L/3$.

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b) Bob leaves Alice behind on earth and travels off to the nearest star 4 light years away and returns. (A light year is the distance light travels in one year) On his return, Bob finds that the total trip according to his clocks is 10 hours less than Alice claims it was. How fast was he travelling? (Assume constant velocity for the trip there and back). (Note– Use 1 year=10^4 hours.)

Twin's paradox: The moving traveller's clock goes slower by $\sqrt{1-v^2/c^2}$. The time that Alice measures the trip as taking is $2L/v$ where v is Bob's velocity. Bob will measure it as $\sqrt{1-v^2/c^2}L/v$ and thus the difference is $2\frac{L}{v}(1-\sqrt{1-v^2/c^2} = 10$ hours. Let us measure distances in light hours, so that c=1. Then L is $8 \cdot 10^4$ light hours, and we have $2\frac{1-\sqrt{1-v^2}}{v} = 2.4 \cdot 10^{-4}$. Expand the square root in a taylor series and keeping only the lowerst order, we get $v = 2.5 \cdot 10^-4$. Note that the whole trip will then take $8\ 10^4/2.5\ 10^{-4} = 3.2\ 10^8$ hours or just over 3000 years. Alice and Bob are very long lived people.

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6.) Light traveling through water at rest travels at a speed of $\frac{1}{n}$ the velocity of light where n is the index of refraction. Assume that the water is traveling at velocity v in the lab in the same direction as the light is traveling. What, to lowest order in v is the velocity of that light according to the lab frame.

Addition of velocities

$$w = \frac{c/n + v}{1 + \frac{vc/n}{c^2}} \tag{46}$$

Doing taylor series expansion get $c/n + v - c/n\frac{vc/n}{c^2} = c/n + v(1-\frac{1}{n^2})$