Physics 407-07
Collapsing Shell

We wish to examine the metric of a collapsing shell in General Relativity. Since we do not yet have Einstein's equation there were be certain features of the solution which will have to taken on faith, or left open, one can determine most of the features simply from knowing the Schwartzschild solution.

Since we are going to want to examine the collapse across the future horizon, we want a set of coordinates in terms of which the future horizon is a regular surface. Since that shell is going to be collapsing in from far away, we also want a set of coordinates for which $r \to \infty$ is well behaved. This suggests using the v r coordinates, where we recall that

$$v = t + r^* = t + r + 2M \ln(\frac{r - 2M}{2M}) \tag{1}$$

where M is an integration constant. We found that if we look at orbits, M is just $\frac{c^2 M}{G}$ is just the Neutonian gravitational mass inside the orbit. Inside the shell since there is no matter inside, we can thus take $M = 0$. Thus the metric inside the shell is

$$d\tau^2 = dv^2 - 2dvdr - r^2 d\Omega^2 \tag{2}$$

where $d\Omega^2$ stands for $d\theta^2 + sin^2(\theta) d\phi^2$.

Outside the shell, we assume that the shell has some mass, and thus the metric should be the Schwarzschild with some non-zero mass. The coordinate r is well defined as the circumference of the sphere over 2π, both inside and outside. However, the coordinate v is not well defined. Let us assume that outside there is some null coordinate $\mathbf{v}$, such that the metric there is

$$d\tau^2 = (1 - \frac{2M}{r}) d\mathbf{v}^2 - 2d\mathbf{v}dr - r^2 d\Omega^2 \tag{3}$$

The shell will be falling toward the black hole by assumption. It will have a trajectory $r(v)$ as seen from the inside and $\mathbf{r}(\mathbf{v})$ as seen from outside. For a given point in time along the shell, which is designated by v $r(v)$ as seen from the inside and $\mathbf{v}$ $\mathbf{r}(\mathbf{v})$ as seen from outside, the circumference must be the same as seen from the inside as from the outside– this is what a thin shell means. Similarly the distance along the inside of the shell from one time to the next must be the same as the distance along the outside. (If we look at Einstein's equations, this is a condition that we not have a singularity at the shell.)

Let us say that we redefine a new outside coordinate called v as well, such that along the surface of the shell, the outside v is the same as the inside. Let us assume that $\mathbf{v} = f(v)$. Then the distance along the inside of the shell will be

$$d\tau = dv^2 - dvdr(v) = dv^2(1 - 2\frac{dr}{dv}) \tag{4}$$

while the distance along the outside is

$$d\tau^2 = (1 - \frac{2M}{\mathbf{r}(\mathbf{v})})d\mathbf{v}^2 - 2d\mathbf{v}d\mathbf{r}(\mathbf{v}) = d\mathbf{v}^2((1 - \frac{2M}{\mathbf{r}(\mathbf{v})}) - 2\frac{d\mathbf{r}}{d\mathbf{v}}) \tag{5}$$

Writing that outside expression in terms of v and recalling that $\mathbf{r}(\mathbf{v}) = \mathbf{r}(f(v)) = r(v)$, we have

$$d\tau^2 = dv^2 f'(v)^2(1 - \frac{2M}{r(v)}) - 2\frac{dr(v(\mathbf{v}))}{d\mathbf{v}}) \tag{6}$$

$$= dv^2 f'(v)^2(1 - \frac{2M}{r(v)}) - 2\frac{dr}{dv}\frac{1}{f'(v)}) \tag{7}$$

Since the v outside is supposed to be the same as the v inside, the distance along the shell between two values of v outside should be the same as the distance along the inside. This gives

$$(1 - 2\frac{dr}{dv}) = f'(v)^2(1 - \frac{2M}{r(v)} - 2\frac{dr}{dv}\frac{1}{f'(v)}) \tag{8}$$

or

$$f'(v) = \frac{\frac{dr}{dv} \pm \sqrt{\frac{dr}{dt}^2 + (1 - \frac{2M}{r})(1 - 2\frac{dr}{dv})}}{1 - 2\frac{M}{r}} \tag{9}$$

Now, for the shell collapsing, $\frac{dr}{dv<0}$. Also, if the shell is going to be timelike, $d\tau^2$ had better be positive, so $-\infty < \frac{dr}{dv} < \frac{1}{2}$. ($\frac{dr}{dv} = -\infty$ corresponds to the shell travelling along the $v = \mathit{const}$ curve, ie is an ingoing light shell.)

We want $f'(v)$ to be positive, since increasing time inside should be equal to increasing time outside. Ie, if $\mathbf{v}$ increases, so should v. This says that we need to take the positive sign of the square root. Note that near infinity, the argument inside the square root becomes $(1-\frac{dr}{dv})^2$ and we have $f'(v)=1$. Ie, near infinity, when the shell is very large, the v coordinate is exactly the same as the $\mathbf{v}$ coordinate.

Now let us look near $r(v)=2M$. Expanding the square root in a taylor series around $r=2M$ we have

$$f'(v) \approx \frac{\frac{dr}{dt} + \left|\frac{dr}{dv}\right| \left(1 + \frac{(1-\frac{2M}{r})(1-2\frac{dr}{dt})}{\frac{dr}{dv}^2}\right)}{1-\frac{2M}{r}} = \frac{1}{2}\frac{(1-2r(v))}{|\frac{dr}{dv}|} \tag{10}$$

Ie, as long as the particle does not stop as it gets to $r=2M$, $f'(v)$ is a perfectly regular function. Ie, there is nothing about the shell as it goes through $r=2M$, as long as it does not stop there, that is at all unusual. Inside the spacetime is Mikowski space, outside it is the massive Schwarzschild spacetime. This is essentially independent of the nature of the function $r(v)$.

Note that inside $r=2M$ we get an additional condition, that the argument of the square root has to be positive. This gives a condition on $r(v)$i if $r(v)<2M$, namely that

$$1-2M\frac{}{r}-\sqrt{-(1-\frac{2M}{r})\frac{2M}{r}} < \frac{dr}{dv} < 1-\frac{2M}{r}+\sqrt{-(1-\frac{2M}{r})\frac{2M}{r}} \tag{11}$$

Note that $r(v)$ is forced to go to zero in a finite v if $r(v)<2M$.

Thus we have in this example that a collapsing shell can go through $r=2M$ with nothing unsual happening there.

Alternative:

Write the equation of the location of the surface r as a function of τ the proper time along the surface. In order for the surface to be a thin shell the equation of motion of the radius (defined as the circumference of the spherical surfaces) must be the same as function of τ, the proper time along the surface, on both sides, assuming one chooses the origin $\tau=0$ appropriately on both sides.

Now find $v(\tau)$ and $\mathbf{v}(\tau)$ on the two sides. Define a new null coordinate V such that $V(\tau)=\tau$ along the surface.

Example:

The metric inside is

$$d\tau^2 = dv^2 - 2dvdr - r^2 d\Omega^2 \tag{12}$$

Let the equation for the surface of the star be given by $r = R(\tau)$ where τ is the proper time along the curve followed by the shell. Defining $\dot{R} = \frac{dR}{d\tau}$. Since we assume that the shell is collapsing, $R(\tau)$ deceases for increasing τ. Let us assume that $\tilde{v}(\tau)$ is the equation for the coordinate v along that surface as a function of τ and $\mathbf{v} = \tilde{\mathbf{v}}(\tau)$ is the equation for the null coordinate outside. We have inside that

$$\frac{d\tilde{v}}{d\tau} = \dot{R} + \sqrt{\dot{R}^2 + 1} \tag{13}$$

which is regular and positive.

Inside we have

$$\frac{d\tilde{\mathbf{v}}}{d\tau} = \frac{\dot{R} + \sqrt{\dot{R}^2 + 1 - \frac{eM}{R}}}{1 - \frac{2M}{R}}$$

which again is regular at $R(\tau) = 2M$ with value

$$\frac{d\mathbf{v}}{dt} \approx \frac{1}{2\dot{R}} \tag{14}$$

Note in order that the curve be timelike, we require that

$$\dot{r}^2 + 1 - \frac{2M}{r} > 0$$

which requires that as $R \to 0$, $\dot{R} \to \infty$. Ie the radial proper velocity must go infinity (r go to zero) faster than $\frac{2}{3(-\tau)^{1/3}}(3M)^{2/3}$.

Now, at $r = R(\tau)$ we want to define the new null coordinate coordinate V such that $V(\tau) = \tau$. Then inside, $v = \tilde{v}(V)$, and outside $\mathbf{v} = \tilde{\mathbf{v}}(V)$. Thus the metric becomes

Inside:

$$d\tau^2 = \frac{d\tilde{v}(V)}{dV}^2 dV^2 - 2\frac{d\tilde{v}(V)}{dV} dV dr - r^2 d\Omega^2 \tag{15}$$

Outside:

$$d\tau^2 = \frac{d\tilde{\mathbf{v}}(V)}{dV}^2 (1 - \frac{2M}{r}) dV^2 - 2\frac{d\tilde{v}(V)}{dV} dV dr - r^2 d\Omega^2 \tag{16}$$

Along the path $r = R(V)$ the two metrics are the same $(dV^2 - R(V)d\Omega^2)$. since by construction along that path, V is just the proper time.

For $v > 0$ we can take $V = \mathbf{v}$.

Collapsing cold star.

For a cold star, there is no heat pressure to hold it up. Then why does a body like the earth not collapse to zero? The answer is the Pauli exclusion principle and quantum mechanics. The following is a very hand waving argument which however gets the answer and the physics right. Let us first operate in the low velocity, low gravity, Newtonian limit, and we will be doing order of magnitude estimates.

For a single electron, with mass m_e, if we confine the electron into a small box with a side of length Δx, then it must have an uncertainty of momentum of at least $\Delta p = \frac{\hbar}{2\Delta x}$. The kinetic energy of each electron is then $\frac{p^2}{2m_e} = \frac{\hbar^2}{2m_e\Delta x^2}$. By the Pauli exclusion principle, if the body of volume V has N_e electrons, and each electron is confined to a box of dimension Δx the total energy will be

$$E_{KE} = \frac{N_e\hbar^2}{\Delta x^2 2m_e} \tag{17}$$

But each electron will be confined to volume of size V/N_e, or of dimension $L/N^{1/3}$ where L is the dimension of the system. Now cosider the electrons in the earth. Because of the Pauli exclusion principle, each electron is essentially confined to a volume whose size is $\frac{V}{N_e} = \frac{4\pi R^3}{3N_e}$ or within a little box of dimension $\Delta x \approx \frac{2R}{N_e^{1/3}}$ Thus the total kinetic energy of the electrons in the earth would be

$$E_{KE} = \frac{N_e^{5/3}\hbar^2}{32m_e R^2} \tag{18}$$

Assuming that there approximately as many nucleons, N_n in the earth as electrons, the gravitiational potential energy is approx

$$\frac{E_G = -\frac{GM^2}{2R} = -GN_n^2 m_n^2}{2R} \tag{19}$$

which gives a total energy of

$$\frac{E = \frac{N_e^{5/3}\hbar^2}{32m_e R^2} - GN_n^2 m_n^2}{2R} \tag{20}$$

Now the earth's radius will adjust itself to minimize this energy, giving us

$$R = \frac{N_e^{5/3}}{N_n^2}\frac{\hbar^2}{8Gm_e m_n^2} \tag{21}$$

(Plugging in the values for the earth, where $N_e \approx N_n = M_e/m_n = \frac{6\ 10^{24}}{1.66\ 10^{-27}kg} = 3.6\ 10^{51}$, $G = 6.67\ 10^{-}11$, $\hbar = 10^{-34}m^2Kg/s$ we get very close ($5\ 10^7m$) to the radius of the earth. Considering our neglect of various factors (exactly what the distribution is of the mass inside the earth, that the nuclei help bind the electrons into slightly tigher orbits than is given by this, etc) this is good agreement.

Now, as N becomes larger and larger, $\Delta x \approx R/N^{1/3}$ becomes smaller and smaller, and eventually Δp becomes relativistic. Also, the kinetic energy of the electrons becomes comparable to the difference in binding energy between the proton plus electron and neutron. At that point the proton swallows the electrons, and all one has left are neutrons.

If the electrons really are relativistic, then $E = pc$,not $E = p^2/2m$ and the kinetic energy becomes

$$\frac{E_{KE} = N_e^{4/3}\hbar c}{4R - \frac{GM^2}{R}}$$

Note that both terms go as 1/R and thus there is no minimum. If the first term is larger than the second, R will increase until it is large enough that the electrons become non-relativistic and one reaches the R of the previous analysis. If

$$N_e^{4/3}\hbar c > 2GM^2 = 2GN_n^2 m_n^2 \tag{22}$$

then the system will keep collapsing until the electon energy becomes comparable to the rest mass energy of the nucleons.

However, long before that, the protons and electrons will inverse beta decay to neutrons. At this point the electron degeneracy pressure becomes irrelevant, and something else must hold up the system and that is the neutron degeneracy pressure. HOwever in either case we get a maximum mass of an object which can support itself by electron degeneracy pressure, Assuming $N_e = \frac{1}{2}N_n$, we get

$$M_{max} = (\frac{\hbar c}{Gm_n^{4/3}})^{3/2} \tag{23}$$

The term $\frac{\hbar c}{G}$ has units of mass squared and is called the Plank mass squared, and the Plank mass is approximately 2.2 $10^{-8} kg$. Thus,

$$M_{max} \approx m_n \frac{m_P^3}{m_n^3} \tag{24}$$

If we took this literally, it would give us a maximum mass of about 2 solar masses. Note that this does not depend on the mass of the electron. In actual fact it does weakly depend on the mass of the electron and the $m_n - m_p - m_e$ neutron binding energy.

Now the neutrons behave the same way– since they are femions, they too are confined to a volume of dimension $\frac{R}{N^{1/3}}$ and have a kinetic energy, and the above analysis goes through with m_e replaced by m_n and N_e replaced by N_n.. There are now two ways of looking at the problem.

a) eventually R becomes smaller than $\frac{GM}{c^2}$. This occurs when

$$\frac{N^{-1/3}\hbar^2}{G m_n m_n^2} \approx G N m_n / c^2 \tag{25}$$

or

$$M = N m_n \approx (\frac{\hbar^2 c^2}{G^2 m_n^{8/3}})^{3/4} \tag{26}$$

Plugging in the values for m_n and the constants of nature, this is very near two solar masses. Ie, a star larger than about one solar mass cannot support itself via the degeneracy pressure of its neutrons, and must collapse. Detailed calculations show that the exact limit depends on the equation of state of nuclear matter in the center of the neutron star, the degree or rotation of the star, but in no case has a value larger than six times of the mass of the sun been found.

Since may stars have much larger masses than this, and since large stars burn up their nuclear fuel faster than lower mass stars, this implies that there must be many black holes in the universe.

If the gas of neutrons is hot (or of electrons and nuclei) then the additional radiation pressure can help support the star, but stars eventually cool, and when they do, and their mass is larger than a few solar masses, they must collapse all the way to r=0.

b) When the neutrons become relativistic, the kinetic energy becomes $E_{KE} = Npc = N\frac{\hbar c}{\Delta x} = N^{2/3}\frac{\hbar c}{R}$ while the gravitational potential energy depends on the energy not the rest mass of the neutrons. Thus it is

$$E_G = -G\frac{(NE_{KE})^2}{Rc^4} = -G\frac{N^{4/3}\hbar^2}{c^2R^3} \tag{27}$$

with the total energy being

$$E = N^{2/3}\frac{\hbar c}{R} - G\frac{N^{4/3}\hbar^2 c^2}{R^3} \tag{28}$$

This has no mimimum. It has a maximum at $R \approx \frac{GM}{c^2}$ where $M = \frac{E_{KE}}{c^2}$. So this relativistic analysis leads to the same conclusionas above. Ie, once N is large enough that the neutrons become relativistic, the system has no stable equilibrium, and must collapse to a black hole.

Both arguments lead to the conclusion that if the mass is larger than a few solar masses, it is impossible for the cold star to support itself via the Degeneracy pressure, and the end result must be a black hole. The collapse of the matter must proceed to all of the matter having been crushed to r=0, zero volume, and infinite density.

Various more exact calculations, using both General relativity for the theory of gravity and taking account of the neutron-neutron interactions (which can give additional pressures), say that the maximum mass of a neutron star is from about 3-6 solar masses, depending on exactly which equation of state is assumed.

So, for a stellar core less than around one solar mass, the end point of evolution is a white dwarf, held up by its electrons. For a mass between about 1 solar mass and 3-6 solar masses it is a neutron star, and for greatr than 6 solar masses, the only possible endpoint must be eternal collapse.