Physics 407-07

Gravity waves and Linear gravity

The geodesic deviation equation

$$\frac{D^2X^i}{D\tau^2} = -R^i{}_{jkl}u^k u^l X^l \tag{1}$$

suggests that if we regard particles travelling under the influence of gravity are simply travelling on straight lines in some metric spacetime, then the geodesic deviation equation corresponds to the tidal motion of particles in a gravitational field. If we assume that u^k are very slow motion timelike vector, $u^t \approx 1$ and the other components approx 0, and the metric is roughly of the form $1 + 2\phi$, then the distance from one particle to the next are approx

$$\frac{d^2X^i}{dt^2} \approx -R^i{}_{ttj}X^j \approx \partial_i\partial_j\phi X^j \tag{2}$$

This suggests that it is trace of $R^i{}_{jkl}$– namely $R_{jk} = -R^i{}_{jki} = R^i{}_{jik}$ is the equivalent of $\nabla^2\phi$, the equation of motion of the Newtonian potential.

Let us look at the case where the the metric is almost the flat Minkowski metric.

$$\eta_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{3}$$

so that

$$g_{ij} = \eta_{ij} + h_{ij} \tag{4}$$

where h_{ij} is small (In the case of the earth the size of h is about or less than 10^{-9} which is very small). Writing all terms to only first order in h, we find

$$R_{ijkl} = \frac{1}{2}(-\partial_l\partial_j h_{ik} + \partial_k\partial_j h_{il} - \partial_k\partial_i h_{jl} + \partial_l\partial_i h_{jk}) \tag{5}$$

(Since only the h_{ij} part of g_{ij} has a non-zero derivative, the Christofel symbols also only contain first order in h terms. Thus terms in the curvature which are of the form $\Gamma\Gamma$ would be of second order and are neglected.) Then

$$R_{ij} = \frac{1}{2}(-\Box h_{ij} - \partial_i\partial_j h + \partial_i\partial_k h^k{}_j + \partial_j\partial_k h^k{}_i) \tag{6}$$

where

$$\Box = \eta^{ij}\partial_i\partial_j \tag{7}$$
$$h = \eta^{ij}h_{ij} \tag{8}$$
$$h^i{}_j = \eta^{ik}h_{kj} \tag{9}$$

If we define

$$\bar{h}_{ij} = h_{ij} - \frac{1}{2}h\eta_{ij} \tag{10}$$

then

$$R_{ij} \approx \frac{1}{2}(-\Box h_{ij} + \partial_i\partial_k\bar{h}^k_j + \partial_j\partial_k\bar{h}^k_i) \tag{11}$$

Now, we can define a new matrix

$$G_{ij} = R_{ij} - \frac{1}{2}g_{ij}g^{kl}R_{kl} \tag{12}$$

we have

$$G_{ij} \approx \frac{1}{2}(-\Box\bar{h}_{ij} + \partial_i\partial_k\bar{h}^k_j + \partial_j\partial_k\bar{h}^k_i - \eta_{ij}\partial_l\partial_k\bar{h}^{kl}) \tag{13}$$

Ie, we have defined G purely in terms of $\bar{h}$

Note that

$$\partial_j G_i{}^j \approx \frac{1}{2}(-\Box\partial_j\bar{h}^j_i + \partial_i\partial_k\partial_j\bar{h}^{jk} + \Box\partial_j\bar{h}^j_i - \partial_i\partial_k\partial_j\bar{h}^{jk}) \tag{14}$$
$$= 0 \tag{15}$$

for any h_{ij}.

The Einstein equations become

$$G_{ij} = 0 \tag{16}$$

in the absense of matter sources.

Now, the coordinates are completely arbitrary. We can simplify the equations if we choose the coordinates approriately. In particular we can choose the coordinates so that the metric obeys

$$\partial_j\bar{h}^j_i = 0 \tag{17}$$

To show this, let us first look at how the metric changes under a coordinate transformation.

$$g_{ij} = \frac{\partial \tilde{x}^k}{\partial x^i} \frac{\partial \tilde{x}^l}{\partial x^j} \tilde{g}(\tilde{x}(x)) \tag{18}$$

Now lets assume that we are going to only make small coordinate transformations. Ie,

$$\tilde{x}^i = x^i + \zeta^i(x) \tag{19}$$

where we assume that ζ are all very small (and of the same order at most as the components of h.

Then

$$\frac{\partial \tilde{x}^k}{\partial x^i} \approx \delta_i^k + \partial_i \zeta^k \tag{20}$$

Now, let us assume that $g_{ij} = \eta_{ij} + h_{ij}$, $\tilde{g}_{ij} = \eta_{ij} + \tilde{h}_{ij}$, and that ζ is of the same order as h (ie, terms which have a ζ and and h are discarded. We find

$$h_{ij} = \tilde{h}_{ij} + \partial_i \zeta_j + \partial_j \zeta_i \tag{21}$$

where $\zeta_i = \eta_{ik} \zeta^k$. Or,

$$\tilde{h}_{ij} = h_{ij} - (\partial_i \zeta_j + \partial_j \zeta_i) \tag{22}$$

We also then have

$$\tilde{\bar{h}}_{ij} = \bar{h}_{ij} - (\partial_i \zeta_j + \partial_j \zeta_i) + \eta_{ij} \partial_k \zeta^k \tag{23}$$

Furthermore, we have

$$\partial_k \tilde{\bar{h}}_i^k = \partial_k \bar{h}_i^k - \Box \zeta_i \tag{24}$$

If we choose

$$\Box \zeta_i = \partial_k \bar{h}_i^k \tag{25}$$

then $\partial_k \tilde{\bar{h}}_i^k = 0$. That equation of ζ can always be solved. Ie, we can always choose our coordinates so that the divergence of $\bar{h}$ is zero.

If this is true, then

$$G_{ij} = \frac{1}{2}(-\Box \bar{h}_{ij}) = 0 \tag{26}$$

Ie, we can always chooose the coordinates such that $\Box h_{ij} = 0$ and $\partial_k \bar{h}_i^k = 0$. These second set of equations are essentially equations which determine the coordinates in which we are trying to solve the equations. They are coordinate conditions.

Let us assume that we have a solution to these equations. There are still additional coordinate transformations we can apply. Let us assume that we have solutions to the equations

$$\Box \bar{h}_{ij} = 0$$

$$\partial_j \bar{h}_i^j = 0$$

Now choose the the new coordinates ζ^i in the following way

$$\zeta_t = \frac{1}{2}\int h_{tt} dt \tag{27}$$

$$\zeta_b = \int h_{tb} dt - \frac{1}{2}\int\int \partial_b h_{tt} d^2 t \tag{28}$$

We have

$$\tilde{h}_{tt} = h_{tt} - 2\partial_t \zeta_t = 0 \tag{29}$$

$$\tilde{h}_{ta} = h_{ta} - \partial_t \zeta_a - \partial_a \zeta_t = 0 \tag{30}$$

$$\tilde{h} = h - 2\partial_t \zeta_t + 2\partial_a \zeta_a \tag{31}$$

$$= h_{tt} - h_{aa} - 2(\frac{1}{2}h_{tt}) + 2(\int \sum_a \partial_a h_{ta} dt - \frac{1}{2}\int\int \sum_a \partial_a \partial_a h_{tt} d^2 t) \tag{32}$$

$$= -h_{aa} + 2\int \partial_t \bar{h}_{tt} - h_t t = 0 \tag{33}$$

where we have used the fact that

$$\bar{h}_{tb} = h_{tb} \tag{34}$$

$$0 = \partial_t \bar{h}_t^j = \partial_t \bar{h}_{tt} - \sum_a \partial_a h_{ta} \tag{35}$$

$$\bar{h}_{tt} = h_{tt} - \frac{1}{2}(h_{tt} - \sum_a h_{aa}) = \frac{1}{2}(h_{tt} + \sum_a h_{aa} \tag{36}$$

and the fact that

$$\Box h_{ij} = \partial_t^2 h_{ij} - \sum_a \partial_a^2 h_{ij} = 0 \tag{37}$$

which means that

$$\Box \zeta_i = 0 \tag{38}$$

since they depend on components of teh tensor h_{ij}.

Ie, given a solution to the equations, one can always change the coordinates so that $h_{tj} = 0$ and $h = \eta^{ij} h_{ij} = 0$. The coordinate conditions $partial_j \bar{h}_i^j = 0$ is called the transverse coordinate condition. (This can be generalised to a coordinate condition on the full metric $\partial_j(\sqrt{|g|} g^{ij}) = 0$ which is called the harmonic coordinate condition, since this corresponds to the condition that the four coordinates obey the condition that

$$\Box x^k = \frac{1}{\sqrt{|g|}} \partial_i \sqrt{|g|} g^{ij} \partial_j x^k = 0 \tag{39}$$

which has been found to be useful in solving the full non-linear Einstein equations.)

The condition that $h = \eta^{ij} h_{ij} = 0$ is called the traceless condition.

Note that these further reductions that the time components and the trace is zero requires that the equations obey the empty space equations. If there is a right hand side to the Einstein equations, we cannot in general make the reduction.

Now, let us assume that we have found a solution to the empty space equation, and that the solution goes as

$$h_{ij} = D_{ij} e^{i\omega(t-z)}$$

where the D_{ij} are constants. This is a plane wave solution travelling in the z direction with velocity of light (since in our units $c = 1$.) This certainly obeys

$$\Box h_{ij} = \partial_t^2 h_{ij} - \partial_x^2 h_{ij} - \partial_y^2 h_{ij} - \partial_z^2 h_{ij} = -\omega^2 h_{ij} - 0 - 0 + \omega^2 h_{ij} = 0$$

Choosing $D_{tj} = 0$ for all j gives the additional coordinate solution. and $\sum_a D_{aa} = 0$ impliments the traceless condition (since $D_{tt} = 0$ already). Finally the "transverse" condition gives $\partial_i {D_a}^i e^{i\omega(t-z)} = 0$ gives $D_{az} = 0$ for all

a. Thus the only terms left are Dxx, D_{yy}, $D_{xy} = D_{yx}$. But the tracefree condition requires that $D_{xx} + D_{yy} = 0$ so we are finally left with only two independent non-zero terms. $D_{xx} = -D_{yy}$ and $D_{xy} = D_{yx}$. These are the two polarizations of the gravity wave.

If we have $D_{xx} = D_{yy} = 0$, and we rotate the coordinates through 45^o, so that $\tilde{x} = (x+y)/\sqrt{2}$ and $\tilde{y} = (x-y)]\sqrt{2}$, we then get

$$\tilde{D}_{xx} = -\tilde{D}_{yy} = D_{xy}$$

Ie, the D_{xy} "cross" polarisation is the same as the other rotated by 45^o.

What do these gravity waves represent? They represent travelling disturbances in distances. The distances in the xy plane change as the metric travels by. These changes in distances can be measured. For example, if one has a solid object, as the distances between the atoms change, the atoms exert forces on each other. Those forces start the atoms moving, and that motion will in general continue after the gravity wave has gone by. Ie, the body will be put into oscillation by the passage of the wave, and that oscillation can be measured. Unfortunately, estimates of the size of the D place it at the 10^{-22} level so that one has to measure oscillations in the body of fractional order of 10^{-22}. This is very very very small. These changes in distance can also be measured directly via a laser interferometer. Since for example if the wave has polarization $D_{xx} - = D_{yy} \neq = 0$, distances in the x direction change in the opposite amount from distances in the y direction, if one sets up an interferometer to compare the x and y lengths of the arms by looking at the interference pattern in an interferometer with its two arms aligned with the x and y directions, one can in principle detect these changes in length. Again, the smallness of the changes makes this a challenge. The current proposal has interferometers with the arm lengths of the order of 4 km, and with the light reflected back and forth about 100 times, making the effective lengths about 400 km. But a 10^{-22} fractional length change is still only $4 \cdot 10^{-17}$m or 10^{-10} of the wavelength of the light. Ie, one has to detect shifts in the interference pattern of one part in 10^{10}. This means one needs extremely high intensities in the interfeeometer arms (kilowatts) without that high energy vapourising the mirrors, and their coatings (or more seriously heating them and changing the distances by the expansion of the coatings). One also needs EXTREMELY good seismic isolation (able to damp out the seizmic vibration amplitude by up to 10^{15}. Two such interferometers exist in the US, one in Hanford Wa, and one in Livingston,Louisiana. While their current sensitivity is not good enough to detect sources believed to exist with a sufficiently high rate of occurence (greater than one a year) the LIGO upgrade expected in the next few years should be sensitive enough to detect gravity waves in the band from 10-1000 Hz.

A proposal has high priority at NASA to place an interferometer in space, with arm lengths of 5,000,000 km called LISA (Laser Interferometer in Space Antenna) to detect very low frequency gravity waves (10^{-4} Hz to 10^{-1} Hz)

0.1 Sources

If $G_{ij} = 0$ is the source free equations, then the source equations must have some tensor on the right hand side. Since the linear equations $\partial_j G_i{}^j = 0$, the right hand side must also be something which obeys sucha a conservation law. It must forthermore (since Newton taught us that mass is the source of gravity) be something that has something to do with the mass or the energy of the system. The conservation of energy can be written as

$$\partial_t e + \sum_a \partial_a (e v^a) = 0$$

where e is the energy density, and ev^a is the energy current. This equation (just like the electrodynamic equation for conservation of charge

$$\partial_t \rho + \nabla \cdot \vec{J} = 0$$

Implies that the only way that the energy within a region can change is by the flow on energy into the region from outside. Similarly, we can write a similar equation for each component of momentum.

$$\partial_t P_a + \nabla \cdot J_{P_a} = 0 \tag{40}$$

The momentum flow into a region could be due to flows of momentum or due to forces acting on the volume (stresses). These equations can be combined into the Stress energy tensor T_i^j where $T_t^t = e$, $T_t{}^a = ev^a$, $T_a{}^t = p_a$ and $T_a{}^b$ is the so called stress tensor– the force with component a across the surface with direction component b.

Ie, if we look at

$$\partial_t \int T^{it} d^3x = \int \partial_t T^{it} d^3x = -\int \sum_a \partial_a T^{ia} = 0$$

Ie, $\int T^{it} d^3$ is a conserved quantity, and the the four conservations laws we know are conservation of energy and conservation of momentum. This tensor is therefor an object which encodes the energy and momemtum tensors, and the way in which energy and momentum flow or change in a small volume.

Let us take the simplest conserved energy momentum tensor, namely $T_{tt} = M\delta^3(x)$. $delta^3(x)$ is the "function (actually distribution) such that $\int \delta^3(x) d^3x = 1$, and which is zero everywhere outside of $x = y = z = 0$. The total energy is thus $\int T^{tt} d^x = M$.

The conservation of the tensor then implies that $M = const$, ie M cannot be a function of time. The solution of Einstein's equations

$$\frac{1}{2}(-\Box \bar{h}_{ij} = \kappa T_{ij} \tag{41}$$

and assuming a time independent solution, gives

$$\frac{1}{2}(\nabla^2 \bar{h}_{tt}) = \kappa T_{tt} = \kappa M \delta^3(x) \tag{42}$$

or

$$\bar{h}_{tt} = 2\kappa M \frac{1}{4\pi r} \tag{43}$$

with all other components of $\bar{h}_{ij}$ zero. Now, $\bar{h}_t t = h_{tt} - \frac{1}{2}\eta^{kl} h_{kl}$, and $0 = \bar{h}_{xx} + \frac{1}{2}\eta^{kl} h_{kl}$, and similarly for h_{yy} and $h_z z$ Since $\eta^{kl} h_{kl} = h_{tt} - h_{xx} - h_{yy} - h_{zz}$, adding up the xx yy zz terms, we have $0 = \frac{1}{2}(h_{tt} + h_{xx} + h_{yy} + h_{zz}$ from which we find

$$\bar{h}_{tt} = 2h_{tt} = 2\kappa M \frac{1}{4\pi r} \tag{44}$$

$$h_{xx} = h_{yy} = h_{zz} = \frac{1}{2}\bar{h}_{tt} = 2\kappa M \frac{1}{4\pi r} \tag{45}$$

Since Einstein argued that $g_{tt} = 1 + 2\phi_{Newtonian}$ and $\phi_{Newtonian} = -\frac{GM}{r}$ we must have that

$$\kappa = -8\pi G \tag{46}$$

Thus Einstein equations become

$$G_{ij} = -8\pi G T_{ij} \tag{47}$$

and the equation for a pure energy distribution with negligible internal stresses is

$$h_{tt} = h_{xx} = h_{yy} = h_{zz} = -2G\frac{M}{r} \tag{48}$$

To compare with the Schwartzschild metric, we need to change the r coordinate. The flat metric is $d\rho^2 + \rho^2 d\theta^2 + \rho^2 sin^2(\theta) d\phi^2$, so we need to change the coordinate from the Schwartzschild r to a new ρ. Assuming r is a function of ρ, we get $dr = \frac{dr}{d\rho} d\rho$. and $r^2 = \frac{r^2}{\rho^2}\rho^2$, If we choose $\frac{1}{1-\frac{2M}{r}}\left(\frac{dr}{d\rho}\right)^2 = \frac{r^2}{\rho^2}$, we can write the Schwartzschild metric as

$$d\tau^2 = (1 - \frac{2M}{r})dt^2 - \frac{r^2}{\rho^2}(d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2(\theta) d\phi^2) \tag{49}$$

and taking $z = r\cos(\theta)$, $x = \rho \sin(\theta)\cos(\phi)$, $y = \rho\sin(\theta)\sin(\phi)$, we have

$$d\tau^2 = (1 - \frac{2M}{r})dt^2 - \frac{r^2}{\rho^2}(dx^2 + dy^2 + dz^2) \tag{50}$$

Solving for $r(\rho)$

$$\frac{\int dr}{r\sqrt{\frac{1-2M}{r}} = \int \frac{d\rho}{\rho} = ln(\frac{2\rho}{M})} \tag{51}$$

or

$$r = \rho(1 + \frac{M}{2\rho})^2 \tag{52}$$

$$\frac{r}{\rho} = \left(1 + \frac{M}{2\rho}\right)^2 \tag{53}$$

So,

$$d\tau^2 = \left(\frac{(1 - \frac{M}{2\rho})}{(1 + \frac{M}{2\rho})}\right)^2 dt^2 - (1 + \frac{M}{2\rho})^4 (dx^2 + dy^2 + dz^2) \tag{54}$$

This gives, to lowest order, $h_{tt} = h_{xx} = h_{yy} = h_{zz} = -\frac{2M}{\rho}$ where $\rho = \sqrt{x^2 + y^2 + z^2}$ which is just the linearized solution.

Note again how the form of a solution can change significantly by just changing the coordinates. Also note that for **this** definition of the radial coordinate (which is not the circumference of the spheres), the horizon is at $\rho = \frac{M}{2}$. Ie, always beware of any claims about the something happening at a certain coordinate. Always find out what that coordinate actually means. Note that $\rho = M/2$ means that the circumference over 2π is $(1 + \frac{M}{2\rho})^2\rho = (1+1)^2\frac{M}{2} = 2M$ as it had better be. What does this coordinate ρ mean? It is the coordinate such that the spatial metric is confomally flat (ie is the flat metric times some overall function.)

Again this is one of the biggest problems with GR. The coordinate dependence makes it very hard to decide what the physics is.