Physics 407-07
Lie derivative

Lie Derivative

In addition to the so called parallel or covariant derivative, there is also an additional concept called the Lie derivative. This derivative is more primative than the covariant derivative in that it assumes less structure on the spacetime.

Assume that we have a series of curves which fill the spacetime. Ie, through each point in the spacetime, there exists a curve from that series of curves, going through that point. We can now use these series of curves to slide the spacetime over itself and to slide any structures on the spacetime over itself. Let us designate the curve from this series of curves going through the point p to be designated by $\gamma_p(\lambda)$ and let the value of the parameter lambda designating the point p to be given by λ_p. Ie, $\gamma_p(\lambda_p) = p$. Now consider the point in the spacetime designated by $\gamma_p(\lambda_p + \mu)$. This will be a new point in the spacetime, near the point p. Let the tangent vector to this curve at p be $\frac{\partial}{\gamma_p}$.

Now consider a function $f(p)$. Define the Lie derivative of the function, designated by

$$\pounds_{\frac{\partial}{\partial\gamma_p}} f = \lim_{\epsilon\to 0} \frac{f(\gamma_p(\lambda_p+\epsilon)) - f(p)}{\epsilon} \tag{1}$$

We note that this is just the derivative of f along the curve γ_p and thus this is just

$$\pounds_{\frac{\partial}{\partial\gamma_p}} f = \frac{\partial}{\partial\gamma_p}^A (df)_A \tag{2}$$

or in coordinates,

$$\pounds_{\frac{\partial}{\partial\gamma_p}} f = \eta^i \partial_i f \tag{3}$$

where we define

$$\frac{\partial}{\partial\gamma_p}^A = \eta^i \frac{\partial}{\partial x^i}^A \tag{4}$$

Now, let us consider the derivative of the cotangent vector defined by the function f. Ie, we want to define the derivative of the cotangent vector $\pounds_{\frac{\partial}{\partial\gamma_p}} df_A$. We do this by subtracting the cotangent vector defined by the dragged function

$$\tilde{f}_\epsilon(p) = f(\gamma_p(\lambda_p - \epsilon)) \tag{5}$$

. We now have the two cotangent vectors df_A and $(d\tilde{f}_\epsilon)_A$ defined at the point p. We can now define the derivative by

$$\pounds_{\frac{\partial}{\partial\gamma_p}{}^A} df_B = \lim_{\epsilon \to 0} \frac{df_B(p) - (d\tilde{f}_\epsilon(p))}{\epsilon} \tag{6}$$

Ie, we define this derivative by comparing the cotangent vector at the point p with that dragged to the point p by the action of the set of curves.

Writing this in coordinate form, we have

$$\tilde{f}_\epsilon(x^i(p)) = f(x^i(\gamma_p(\lambda_p - \epsilon))) \approx f(x^i) - \epsilon\eta^j\partial_j f + O(\epsilon^2) \tag{7}$$

The components of the cotangent vector are

$$(d\tilde{f}_\epsilon(p))_i = \partial_i(\tilde{f}_\epsilon(p)) = \partial_i f - \epsilon\eta^j\partial_j f) \tag{8}$$

and the Lie derivative then is

$$\pounds_{\eta^A} = \partial_i\eta^j\partial_j f + \eta^j\partial_j(\partial_i f) \tag{9}$$

Thus for a generic cotangent vector with components U_i we have

$$\pounds_{eta^A} U_i = \eta^j\partial_j U_i + U_j\partial_i\eta^j \tag{10}$$

We can equivalently define the Lie derivative of a tangent vector by noting that $V^A W_A$ is an ordinary function, and thus

$$\pounds_{\eta^A} V^B W_B = \pounds_{\frac{\partial}{\partial\gamma_p}{}^A} V^i W_i \tag{11}$$

$$= (\eta^j\partial_j V^i)W_i + V^i(\eta^j\partial_j W_i) \tag{12}$$

$$= (\eta^j\partial_j V^i - V^j\partial_j\eta^i)(W_i) + V^i(\eta^j\partial_j W_i + W_j\partial_i\eta^j) \tag{13}$$

$$(\pounds_{\frac{\partial}{\partial\gamma_p}{}^A} W_B)_i V^i + W_i((\eta^j\partial_j V^i - V^j\partial_j\eta^i) \tag{14}$$

Thus we define

$$\pounds_{\eta^A} V^i = (\eta^j \partial_j V^i - V^j \partial_j \eta^i) \tag{15}$$

NOte that

$$\pounds_{V^A} U^B + \pounds_{U^A} V^B = 0 \tag{16}$$

The Lie derivative of the metric is given by

$$\pounds_{\eta^A} g_{ij} = \eta^k \partial_k g_{ij} + g_{ik} \partial_j \eta^k + g_{kj} \partial_i \eta^k \tag{17}$$
$$= \eta^k \partial_k g_{ij} + \partial_j \eta_i + \partial_i \eta_j - \eta^k (\partial_i g_{kj} + \partial_j g_{ik}) \tag{18}$$
$$= \partial_j \eta_i + \partial_i \eta_j - 2\eta^k \Gamma_{kij} \tag{19}$$
$$= \partial_j \eta_i + \partial_i \eta_j - 2\eta_k \Gamma^k_{ij} \tag{20}$$

Now, if the metric dragged along the curve is identical to the metric, then this is called a symmetry of the spacetime. This means that if there exists a vector field K^A such that

$$\pounds_{K^A} g_{BC} = 0 \tag{21}$$

then the vector field K^A is a symmetry of the spacetime. Such vectors are called Killing vectors.

A spacetime can contain at most 10 linearly independent Killing vectors.

Consider the Killing equation components

$$\partial_i K_j = \frac{1}{2}(\partial_i K_j - \partial_j K_i) \tag{22}$$
$$+\frac{1}{2}(\partial_i K_j + \partial_j K_i + K_k \partial_i g_{kj} - 2K_k \Gamma^k_{ij}) + K_k \Gamma ij^k \tag{23}$$
$$= \frac{1}{2}(\partial_i K_j - \partial_j K_i) - \frac{1}{2}(K_k \Gamma^k_{ij}) \tag{24}$$

Ie the derivative of K^i in the direction j can be written in terms of the derivative the antisymmetric derivative of K and of the value of K.

We can also write the Killing equation as

$$\partial_i K_j = -\partial_j K_i + K_k \Gamma^k_{ij} \tag{25}$$

Looking at the derivative of the antisymmetric derivative

$$\partial_k(\partial_i K_j - \partial_j K_i) = \partial_i(\partial_k K_j) - \partial_j\partial_k K_i \tag{26}$$
$$= \partial_i\partial_j K_k - \partial_j\partial_i K_k + \partial_i(\Gamma^l_{ki}K_l) - \partial_j(\Gamma^l_{ik}K_l) \tag{27}$$
$$= (\partial_i\Gamma^l_{ki}) - \partial_j\Gamma^l_{ki})K_l + \Gamma^l_{ki}\partial_j K_l - \Gamma^l_{kj}\partial_i K_l \tag{28}$$
$$= \left(\partial_i\Gamma^l_{ki} - \partial_j\Gamma^l_{ki} + \Gamma^m_{ki}\Gamma^l_{jm} - \Gamma^m_{kj}\Gamma^l_{im}\right)K_l \tag{29}$$
$$+\frac{1}{2}\left(\Gamma^l_{ki}(\partial_j K_l - \partial_l K_j) - \Gamma^l_{kj}(\partial_i K_l - \partial_l K_i)\right) \tag{30}$$

Ie, the derivative of the antisymmetric derivative can be expressed in terms of derivatives of the metric times the components of the Killing vector plus derivatives of the metric times components of the antisymmetric derivative of the Killing tensor (since the ordinary derivative can be expressed in terms of the antisymmetric derivative and derivatives of the metric times the components of the Killing vector.). Ie, we have an intial value equation, in which if we specify the 4 components of the Killing vector and the six components of the antisymmetric derivative of the Killing vector at a point, then we can integrate them up along all of the coordinate axes, and everywhere in the spacetime.

It is of course also required that if we integrate up the equations along different paths, we get the same vector. This is what can reduce the number of Killing vectors to less than 10, but there can never be more than 10.

Flat spacetime has 10.

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2 \tag{31}$$
$$K(1)_i = (1,0,0,0) \tag{32}$$
$$K(2)_i = (0,1,0,0) \tag{33}$$
$$K(3)_i = (0,0,1,0) \tag{34}$$
$$K(4) = (0,0,0,1) \tag{35}$$
$$K(5)_i = (x,-t,0,0) \tag{36}$$
$$K(6)_i = (y,0,-t,0) \tag{37}$$
$$K(7)_i = (z,0,0,-t) \tag{38}$$
$$K(8)_i = (0,y,-x,0) \tag{39}$$
$$K(9)_i = (0,z,0,-x) \tag{40}$$
$$K(10)_i = (0,0,z,-y) \tag{41}$$

where the (a, b, c, d) means that the t component is a, the x is b, the y is c and the z is d.

The first four have zero antisymmetric derivatives at $t = x = y = z = 0$, but have non-szero value for one of the components of the Killing vector at that point. The last 6 have zero value for all components at $t = x = y = z = 0$, but have non-zero antisymmetric derivative there.

(NOte that any linear combination of Killing vectors with constant coefficients is also a Killing vector.)