Physics 407-08

Straight lines

Simplified variation principle

In the variational principle I used before, I chose the parameter λ to be arbitrary, but so that λ_0 and λ_1 are fixed at the end points– ie so that the limits in the integration do not depend on the path (do not depend on ϵ). This however came at a price. The variational equations have the ugly terms $\sqrt{\sum_{ij} g_{ij}\partial_\lambda x^i \partial_\lambda x^j}$ in the differential equations. By defining the new parameter s by

$$s = \int_{\lambda_0}^{\lambda} \sqrt{\sum_{ij} g_{ij}\partial_\lambda x^i \partial_\lambda x^j} d\lambda \tag{1}$$

in the final equations of motion, one can greatly simplify the equations. Can one simply start out using the parameter s? The answer is no. As one changes the path, the value of s at the end point (where $\lambda = \lambda_1$) changes. Since the straight line is the shortest distance, for any other choice of path the distance (the integral along the path of ds) changes.

However, one can do something else. Let us choose λ so that **on the straight line**, $\lambda = s$. Ie, on the straight line itself, $\sqrt{\sum_{ij} g_{ij}\partial_\lambda x^i \partial_\lambda x^j} = 1$. Off the straight line, (ie for values of ϵ which do not minimize the length) the parameter λ is not the path length parameter, but maintains the same values, $\lambda_0 = 0$ and $\lambda_1 = L$, at the end points. Thus, in the equations for the solutions derived via variation we can assume that that square root is one. This immediately gives us the simplified equations in terms of the path length parameter s.

In fact we can go further than this. Let us multiply the argument of the integral by some power of that square root. Ie, let us look at the integral

$$N(\epsilon) = \int_{\lambda_1}^{\lambda_2} \sqrt{\sum_{ij} g_{ij}\partial_\lambda x^i \partial_\lambda x^j}^{\,n} d\lambda \tag{2}$$

recalling that all of the x_i are functions of both λ and ϵ, and that the g_{ij} are functions of the coordinates x^i which are functions of λ and ϵ. This is NOT the path length. However, let us also assume that we choose λ so that **on the path which minimizes N**, λ is the same as s.

In taking the derivatives with respect to ϵ, we get

$$\frac{dN}{d\epsilon} = \int_{\lambda_1}^{\lambda_2} n\sqrt{\sum_{ij} g_{ij}\partial_\lambda x^i \partial_\lambda x^j}^{n-1} \partial_\epsilon \sqrt{\sum_{ij} g_{ij}\partial_\lambda x^i \partial_\lambda x^j} d\lambda \tag{3}$$

On the solutions, which are the only place where we are interested in the above, we see, that as long as $n \neq 0$ we can write this as

$$\frac{dN}{d\epsilon} = n\int_{\lambda_1}^{\lambda_2} \frac{d}{d\epsilon}\sqrt{\sum_{ij} g_{ij}\partial_\lambda x^i \partial_\lambda x^j} d\lambda|_{\lambda=s} = n\frac{dL}{d\epsilon} \tag{4}$$

(Note that we cannot replace the square root within the derivative with respect to ϵ with 1, since it depends of values of ϵ which are not at the minimum)

Ie, if we set $\frac{dL}{d\epsilon} = 0$ then the equations we get are identical to those we get if we set $\frac{dN}{d\epsilon} = 0$ (as long as $n \neq 0$). Of course, this does not help much unless we take $n = 2$ In that case we get

$$\frac{dN}{d\epsilon} = \frac{d}{d\epsilon}\int_{\lambda_1}^{\lambda_2} \sum_{ij} g_{ij}\partial_\lambda x^i \partial_\lambda x^j d\lambda \tag{5}$$

but where we MUST remember that λ must be chosen so that on the solution curve, $\lambda = s$. This expression is almost always written as

$$int_{s_1}^{s_2} \sum_{ij} g_{ij}\partial_s x^i \partial_x^j ds \tag{6}$$

If we took this seriously, the argument would be unity, and also the limits would depend on ϵ, and this would be a hopeless thing to vary. But this expression does not mean what it seems to mean. It means, take the parameter λ to be equal to the path length **only on the solution path**. It is a shorthand for the previous equation, an immensely useful shorthand, but also a potentially confusing one if you do not keep its true meaning in mind.

But this variation is vastly simpler in that one does not get those ugly square roots in all of ones equations. That simplification is worth all of the potential confusion. (Well, it is to most practitioners who have gotten used to it. Whether it is worth it to students may well be a different question.)