Physics 407-07
Gravity waves source

The linearized equations for gravity waves

$$G_{ij} = -8\pi T_{ij}$$

(taking Newton's constant G=1), and choosing the harmonic or transverse coordinate condition

$$\partial_j \bar{h}_i^j = 0$$

is

$$-\frac{1}{2}\Box \bar{h}^{ij} = -8\pi T^{ij} \quad (1)$$

Let us choose our stress energy tensor so as to be a generator of bravity waves. In order that the stress-energy tensor be time dependent, we need that the spatial parts be non-zero. Let us take

$$T^{ab} = \partial_t{}^4 D^{ab}(t)\delta^3(\mathbf{x}) \quad (2)$$

where a, b are x, y, z, and $\mathbf{x}$ is all three of x, y, z. That fourth time derivative is there so as to make some of the successive equations easier to write (ie not to have integrals contained in them).

Then we have from the conservation of T^{ij}, ie $\partial_j T^{aj} = 0$ that

$$T^{at} = -\sum_b \partial_t^3 D^{ab}\partial_b \delta^3(\mathbf{x})$$

and from

$$\partial_j T^{tj} = 0$$

$$T^{tt} = \sum_{a,b} \partial_t^2 D^{ab}\partial_a\partial_b \delta^3(\mathbf{x})$$

Now, the solution to the equation

$$\Box\Psi(t, \mathbf{x}; t, \mathbf{x}') = \delta(t - t')\delta^3(\mathbf{x} - \mathbf{x}')$$

is

$$\Psi(t, \mathbf{x}; t, \mathbf{x}') = \frac{1}{4\pi}\frac{\delta(t - t' - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|} \quad (3)$$

where $|\mathbf{x}-\mathbf{x}'| = \sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$, the spatial distance between the two spatial points $\mathbf{x}$ and $\mathbf{x}'$.

Thus we get

$$\bar{h}^{tt} = 4\int \frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\partial_{t'}^4 D(t')\partial_{a'}\partial_{b'}\delta^3(\mathbf{x}')dt'd^3x' \tag{4}$$

where $\partial_{a'} = \frac{\partial}{\partial x'^a}$ Now, if for arbitrary functions f and g, we have

$$\int f(u-u')\partial_{u'}g(u')du' = -\int(\partial_{u'}f(u-u'))g(u') \tag{5}$$

$$= \int \partial_u f(u-u')g(u')du' = \partial_u \int f(u-u')g(u')du' \tag{6}$$

Where on the first line I have done integration by parts, assuming the boundary terms are zero, and on the second line using the fact that

$$\partial_{u'}f(u-u') = -\partial_u f(u-u')$$

Using this both for the 4 time derivatives and the two spatial derivatives, we have

$$\bar{h}^{tt} = 4\sum_{a,b}\partial_t^2\partial_a\partial_b\int\frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}D^{ab}(t')\delta^3(\mathbf{x})dt'd^3x' \tag{7}$$

$$= 4\sum_{a,b}\partial_t^2\partial_a\partial_b\frac{D^{ab}(t-r)}{r} \tag{8}$$

$$= 4\sum_{a,b}\partial_a\partial_b\frac{\partial_t^2 D(t-r)}{r} \tag{9}$$

where $r = |\mathbf{x}| = \sqrt{x^2+y^2+z^2}$. Similarly

$$\bar{h}^{ta} = h^{ta} = -4\sum_b \partial_b\frac{\partial_t^3 D^{ab}(t-r)}{r} \tag{10}$$

$$\bar{h}^{ab} = 4\frac{\partial_t^4 D^{ab}}{r} \tag{11}$$

The contribution of the "trace" part of D^{ab} on $\bar{h}^{ij}$ can be removed by a coordinate transformation. In the next three paragraphs we will show that a purely trace part of D^{ab}, ie $D^{ab} = D\delta^{ab}$ can be removed by a coordinate transformation, so that we can assume that $\sum_a D^{aa} = 0$.

Let us assume that $D^{ab} = D\delta^{ab}$. Then $\bar{h}^{tt} = 4\partial_t^2\nabla^2\frac{D(t-r)}{r}$ and outside the source, since $\bar{h}$ obeys the wave equation which is

$$\partial_t^2\bar{h}^{tt} - \nabla^2\bar{h}^t t = 0 \tag{12}$$

we have

$$\bar{h}^{tt} = 4\partial_t^2\nabla^2(D(t-r)/overr) = 4\frac{\partial_t^4 D(t-r)}{r} \tag{13}$$

Each of the diagonal spatial terms is the same as this. Since,

$$\bar{h}^{ij} = h^{ij} - \frac{1}{2}\eta^{ij}\eta_{kl}h^{kl}$$

we have

$$\eta_{ij}\bar{h}^{ij} = \eta_{ij}h^{ij} - \frac{1}{2}\eta_{ij}\eta^{ij}\eta_{kl}h^{kl} = -\eta_{ij}h^{ij}$$

and

$$h^{ij} = \bar{h}^{ij} - \frac{1}{2}\eta^{ij}\eta_{kl}\bar{h}^{kl} \tag{14}$$

Ie the relation between h and $\bar{h}$ is exactly the same as the relation between $\bar{h}$ and h.

Since all of the terms are the same (

$$\bar{h}^{tt} = \bar{h}^{xx} = \bar{h}^{yy} = \bar{h}^{xx} = 4\partial_t^4\frac{D(t-r)}{r}$$

, we have $\eta_{ij}\bar{h}^{ij} = -2\bar{h}^{tt}$ in this case. Then

$$h^{tt} = 2\bar{h}^{tt} = 8\frac{\partial_t^4 D(t-r)}{r}$$

, and $h^{xx} = h^{yy} = h^{zz} = 0$. Now, choose a coordinate transformation $\zeta^t = 4\frac{\partial_t^3 D(t-r)}{r}$. This will set $\tilde{h}^{tt} = h^{tt} - 2\eta^{tj}\partial_j\zeta^t = 0$. Also,

$$\tilde{h}^{ta} = h^{ta} - \eta^{aj}\partial_a\zeta^t - \eta^{tj}\partial_j\zeta^a = -4\partial_a\frac{\partial_t^3 D(t-r)}{r} - (-1)\partial_a 4\frac{\partial_t^3 D(t-r)}{r} = 0$$

Ie, by a change of only the time coordinate η^t, one can eliminate the solution everywhere outside the source. Since we can, by a coordinate transformation make the h^{ij} look as though we have any trace term we want, we can therefor assume that $\sum_a D^{aa} = 0$.

Thus, let us assume that $\sum_a D^{aa} = 0$. Go back to the full expression with arbitrary D^{ab} but with $\sum_a D^{aa} = 0$.

$$h_{tt} = \bar{h}^{tt} - \frac{1}{2}\eta_{ij}\bar{h}^{ij} = \frac{1}{2}\bar{h}^{tt} \tag{15}$$

$$= 2\left(\sum_{a,b} \partial_a\partial_b \frac{\partial_t^2 D^{ab}}{r}\right) \tag{16}$$

Choosing

$$\zeta^t = \left(\sum_{a,b} \partial_a\partial_b \frac{\partial_t D^{ab}}{r}\right)$$

and and

$$\zeta^a = \left(\sum_{d,c} \partial_a\partial_c\partial_d \frac{D^{cd}(t-r)}{r}\right) - 4\sum_d \partial_d \frac{\partial_t^2 D^{ad}(t-r)}{r} \tag{17}$$

$$\tilde{h}^{tt} = h^{tt} - 2\eta^{tt}\partial_t\zeta^t = 0$$

$$\tilde{h}^{ta} = h^{ta} - \eta^{aa}\partial_a\zeta^t - \eta^{tt}\partial_t\zeta^a = h^{ta} + \partial_a\zeta^t - \partial_t\zeta^a = 0$$

$$\tilde{h}^{ab} = h^{ab} - \eta^{aa}\partial_a\zeta^b - \eta^{bb}\partial_b\zeta^a$$

$$= \bar{h}^{ab} - \frac{1}{2}\eta^{ab}\bar{h}^{tt} + \partial_a\zeta^b + \partial_b\zeta^a = 4\frac{\partial_t^4 D^{ab}(t-r)}{r} + 2\delta^{ab}\sum_{cd}\partial_c\partial_d\frac{\partial_t^2 D}{}$$

$$- 4\sum_d \partial_a\partial_d \frac{\partial_t^2 D^{bd}}{r} + 4\sum_d \partial_b\partial_d \frac{\partial_t^2 D^{da}}{r}$$

Thus the metric has again been reduced to a purely spatial metric by a appropriate coordinate transformation.

If we looks at the temporal part of the stress energy tensor, we find that

$$\int x^a x^b T^{ab} d^3x = \int x^a x^b \partial_t^2 D(t)\partial_c\partial_d\delta^3(\mathbf{x}) \tag{23}$$

$$= \partial_t^2(D^{ab} + D^{ba}) = 2\partial_t^2 D^{ab}(t) \tag{24}$$

Ie, the tensor $2\partial_t^2 D^{ab}(t)$ is just the quadrapole moment of the source of radiation. The trace is the "radial" changes in the energy distribution, and these we found produce no gravitational radiation (ie it can be removed by a coordinate transformation) It is thus the trace free part of the quadrapole moment that produces the gravity waves. Ie, gravitational radiation is just proportional to time derivatives of the quadrapole moment of the sources.

If we look far from the source, and keep only terms which fall off as 1/r, then $\partial_a \frac{D(t-r)}{r} \approx -\frac{x^a}{r}\frac{\partial_t D(t-r)}{r}$ and $\partial_a n^b = \frac{\delta^{ab}}{r} - \frac{n^a n^b}{r^2} \approx 0$ (ie it falls off too fast with r), we find

$$h_{ab} = \frac{2}{r}\partial_t^2 \left(Q^{ab} - n^a \sum_c n^c Q^{cb} - n^b \sum_c Q^{ca} \right. \tag{25}$$

$$\left. + \frac{1}{2}(n^a n^b \sum_{cd} n^c n^d Q^{cd} + \delta^{ab} n^c n^d Q^{cd}) \right) \tag{26}$$

$$= 2\frac{\partial_t^2 Q_\perp^{ab}(t-r)}{r} \tag{27}$$

where $Q_\perp^{ab}$ is the component of the quadrapole moment perpendicular to the direction from the source and trace free. Ie, $\sum_b Q_\perp^{ab} n_b = 0$ where $n^b = x^b/r$.

$$Q_\perp^{ab} = Q^{ab} - \sum_c (Q^{ac} n^c n^b + Q^{bc} n^c n^a) + \frac{1}{2}\sum_{cd}(\sum_{cd} Q^{cd} n^c n^d (n^a n^b + \delta^{ab})) \tag{28}$$

Ie, the radiation is proportional to the second derivative of the quadrapole moment perpendicular to the direction the radiation is travelling in.

The energy carried away in the field can, in this coordinate system, be written as

$$\frac{1}{4}\sum_{ab}\left((\partial_t h_{ab})^2 + \sum_c (\partial_c h_{ab})^2 \right) \tag{29}$$

and thus is proportional to the square of the third derivative of the Quadrapole moment of the source. The quadrapole moment is of the form mL^2, where m is the part of the total mass which is moving around changing the quadrapole moment. This means that the third derivative is of order $\partial_t Q = mL^2/T^3 = \omega m v^3/L$ where v is the velocity of the matter. Thus the loss of energy per unit mass is of the order of $\omega(\frac{m}{M})^2 M v^5/$ or reinserting units, $\omega(\frac{m}{M})^2 \frac{GM}{c^2 L}\frac{v}{c})^5$ Ie, in each period the system sheds a fraction of the mass proportional to the square of the ratio of moving mass over total mass times the ratio of the Schwartzschild radius over the dimension of the system times the velocity over c to the fifth power. Ie, Gravitational radiation is weak. To get a sizeable fraction radiated you need something like two black holes orbiting each other at the last stable orbit, and even then less that 1% of the total mass is radiated as gravitational radiation.