Physics 407-08
Collapse of Stars

What happens when stars run out of fuel? This was a question S. Chandrasekhar asked himself while travelling by ship from India to Cambridge to start his studies there. For hot stars, it is the heat pressure which maintains the star. If the star's radius decreases slightly, this compression increases the temperature of the star– both because of adiabatic heating and because of the increase in nuclear reaction in the start as the density rises. But what happens when the nuclear reactions stop, and the star begins to cool down?

For a small enough star (or for a body like a baseball, the earth, or a star whose mass is up to about one solar mass) the answer is that the so called electron degeneracy pressure. Consider that as the radius of the star decreases, the amount of room available for each electron decreases. Because one cannot have two electrons being in the same state, we can approximate the system by saying that each electron has a volume of approximately

$$V = \frac{R^3}{N_e} \tag{1}$$

(Note that I will be dropping all constants of order unity since these will be just rough estimates) $N_e$ is the total number of electrons. Thus the dimensions of the box in which each electron is confined has a length of $V^{1/3}$, and the momentum of the electrons is then

$$p = \frac{\hbar}{V^{1/3}} \tag{2}$$

If this is non-relativistic, the total energy of these electrons is $N_e p^2/m_e$.

$$E_e = N_e \hbar^2 \frac{N_e^{2/3}}{R^2 m_e} = \hbar^2 \frac{R^2 N_e^{5/3}}{m_e} \tag{3}$$

where I have set $\pi/3 = 1$.

The graviational potential energy is approximately $\frac{GM^2}{R} = G\frac{(N_n m_n)^2}{R}$ where $m_n$ is the mass of a nucleon and $N_n$ the total number of nucleons. Thus for a cold star the approximate total energy is

$$E = \hbar^2 \frac{N_e^{5/3}}{R^2 8 m_e} - \frac{3}{5} G \frac{(N_n m_n)^2}{R} \tag{4}$$

which has a minimum of

$$R_{min} = \frac{\hbar^2}{G}\frac{N_e^{5/3}}{N_n^2}\frac{5}{24 m_e m_n^2} \tag{5}$$

Ie, as long as the electrons are not squeezed into a box so small that they become relativistic, the star has a stable equilibrium radius. Note that as the number of nucleons increases the radius slow decreases, not increases.

This is how the earth, and white dwarf stars hold themselves up against gravity. (in Jupiter, it is still at a temperature higher than 0 so it is primarily held up by thermal pressure).

Since the number of nucleons is approximate the same as the number of electrons, as $N_n$ increases, the cold radius decreases, and the momentum increases of each electron increases, until the electrons become relativistic. Then the energy of the electrons becomes equal to $pc$.

The total Kinetic energy is now

$$E_e = N_e c \frac{\hbar}{V^{1/3}} = N_e^{4/3} \hbar c \frac{1}{R} \tag{6}$$

Both the kinetic energy contribution and the gravitational energy go as $1/R$ which means that the system is unstable if

$$\hbar N_e^{4/3} c < G(N_n m_n)^2 \tag{7}$$

Thus there is a critical number of nucleons and thus of mass at which the system is not relativistically stable, but the collapse proceeds to smaller and smaller radia Taking $N_e = N_n$ we have

$$\frac{M_c = N_n m_n = (\hbar c}{G)^{3/2}\frac{1}{m_n^2}} \tag{8}$$

Now, $(\frac{\hbar c}{G})^{1/2}$ is a quantity of dimensions of mass, and is called teh Planck mass $M_P$ and is about $10^{-5} gm$. Thus the maximum mass which is relativistically stable is given by the incredibly simple relation

$$M_c = \frac{M_P^3}{m_n^2} \tag{9}$$

However, the collapse before the electrons become completely relativistic, the electrons are absorbed by the protons in the nucleus, forming neutrons.

Neutrons are fermions as well, which implies that they also have the same behaviour as the electrons above. One can replace the $m_e$ and $N_e$ by $m_n$ and $N_n$ in the above, to find that there is also a maximum mass size for a star made up entirely of neutrons. Again, if $R$ becomes small enough that the neutrons become relativistic, there is again no minimum. This implies that there is maximum number of nucleons in a star such that the star can support itself by neutron degeneracy pressure. Any star larger than that must either shed off the excess mass or it has no minimum, and must collapse to a black hole.

The maximum mass for the electron degeneracy pressure to support is the so called Chandrasekhar mass and is about 1.4 solar masses. A star which is less massive than this can support itself by electron degeneracy pressure. If the mass is bigger than about .1 times the mass of the sun, this is a white dwarf. If it is about 100 more massive than Jupiter, it is a brown dwarf. If less, it is a planet. The maximum mass for a neutron start is very similar– and in fact depends in detail on the pressure of the nuclear matter. As well as the degeneracy pressure, there are interactions between the nucleons which could increase or decrease the maximum mass.

This means that there must be lots of black holes out there, since there are many stars with masses much larger than the mass of the sun.