Physics 407-09
Assignment 6

1) Show that the vectors K_1^A and K_2^A with components

$$K_1^t = K_1^r = 0 \tag{1}$$
$$K_1^\theta = \cos(\phi) \tag{2}$$
$$K_1^\phi = -\sin(\phi)\cot(\theta) \tag{3}$$

and

$$K_2^t = K_2^r = 0 \tag{4}$$
$$K_2^\theta = \sin(\phi) \tag{5}$$
$$K_2^\phi = \cos(\phi)\cot(\theta) \tag{6}$$

are Killing vectors of the two dimensional metric

$$ds^2 = d\theta^2 + sin(theta)^2 d\phi^2$$

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The equation for a Killing vector is

$$K^k\partial_k g_{ij} + g_{kj}\partial_i K^k + g_{ik}\partial_j K^k \tag{7}$$

(using the summantion convention) Thus

$$K^\theta\partial_\theta g_{ij} + g_{\theta j}\partial_i K^\theta + g_{\phi j}\partial_i K^\phi + g_{i\theta}\partial_j K^\theta + g_{i\phi}\partial_j K^\phi \tag{8}$$

Now, the only derivaties of K are with respect to θ or ϕ, and the only terms in g which are dependent on θ or ϕ is $g_{\phi\phi}$.

Also g is diagonal. The only terms which are nonzero are where $i,\ j$ are $\theta,\ \phi$. (The first term is non=zero only if ij are both ϕ. the second is always zero.Thus, the

$$\theta\theta \tag{9}$$
$$2g_{\theta\theta}\partial_\theta K^\theta = 0 \tag{10}$$
$$\theta\phi \tag{11}$$
$$g_{\phi\phi}\partial_\theta K^\phi + g_{\theta\theta}\partial_\phi K^\theta \tag{12}$$
$$\phi\phi \tag{13}$$
$$K^\theta\partial_\theta g_{\phi\phi} + 2g_{\phi\phi}\partial_\phi K^\phi \tag{14}$$

Each of these is zero.

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Show that the Lie derivative of K_1^A by K_2^A is the third rotational Killing vector whose ϕ component is 1 and others are zero.

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$$(\pounds_{K_1}K_2)^i = K_1^\theta\partial_\theta K_2^i + K_1^\phi\partial_\phi K_2^i - K_2^\theta\partial_\theta K_1^i + K_2^\phi\partial_\phi K_1^i \tag{15}$$

The components are

$$\theta \tag{16}$$

$$K_1^\theta\partial_\theta K_2^\theta + K_1^\phi\partial_\phi K_2^\theta - K_2^\theta\partial_\theta K_1^\theta - K_2^\phi\partial_\phi K_1^\theta \tag{17}$$

$$= K_1^\phi\partial_\phi K_2^\theta - K_2^\phi\partial_\phi K_1^\theta = 0 \tag{18}$$

$$\phi \tag{19}$$

$$K_1^\theta\partial_\theta K_2^\phi + K_1^\phi\partial_\phi K_2^\phi - K_2^\theta\partial_\theta K_1^\phi - K_2^\phi\partial_\phi K_1^\phi \tag{20}$$

$$= cos(\phi)(cos(\phi)\frac{-1}{sin^2(\theta)} + (-sin(\phi))cot^2(\theta)(-sin(\phi)) \tag{21}$$

$$-sin(\phi)(-sin(\phi)\frac{-1}{sin^2(\theta)} + cos^2(\phi)cot^2(\theta) = -1 \tag{22}$$

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2.a) Find the radial geodesic equations for light emitted from r=0 at $t = t_1$ and absorbed at $r = R$ at time $t = t2$ in the standard t, r, θ, ϕ coordinates for the homogeneous and isotropic cosmological spacetimes.

$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2(d\theta^2 + sin(\theta)^2 d\phi^2))$$

Assume $\theta = \pi/2$. Given a divergence $\delta\phi$ in two light rays from the source, what is the spatial distance between them at $t = t_2$ and $r = R$. Assume that the light is emitted uniformly from the star at $r = 0,\ t = t_1$, how does the intensity of the light drop off as a function of R? (If N photons per second are emitted uniformly in direction from the star at time t_1 how many of them will cross a unit area in unit time at $t = t_2$ and $r = R$?)

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$$-2\frac{d}{ds}(a^2r^2sin^2(\theta)\frac{d\phi}{ds}) = 0 \tag{23}$$

$$-2\frac{d}{ds}(a^2r^2\frac{d\theta}{ds}) + 2a^2r^2\sin(\theta)\cos(\theta)(\frac{d\phi}{ds})^2 = 0 \tag{24}$$

$$-2\frac{d}{ds}(\frac{1}{1-kr^2}\frac{dr}{ds} + 2a^2r(\frac{d\theta}{ds}^2 + sin^2(\theta)\frac{d\phi}{ds}^2) = 0 \tag{25}$$

$$2\frac{d^2t}{ds^2} + 2a\frac{da}{dt}(\frac{1}{1-kr^2}\frac{dr}{ds}^2 + r^2\frac{d\theta}{ds}^2 + r^2sin(\theta)^2\frac{d\phi}{ds}^2) \tag{26}$$

The first equation gives

$$\frac{d\phi}{ds} = \frac{l}{a^2r^2} \tag{27}$$

Substituting into the second one, multiplying by a^2r^2, and defining defining $\lambda = \frac{ds}{a^2r^2}$ we have have

$$\frac{d^2\theta}{d\lambda^2} - \frac{l^2 cos(\theta)}{sin(\theta)^3} = 0 \tag{28}$$

$$\text{or multiplying by} \frac{d\theta}{d\lambda} \tag{29}$$

$$\frac{d\theta}{d\lambda}^2 + \frac{l^2}{sin(\theta)^2} = L^2 \tag{30}$$

$$\text{or} \tag{31}$$

$$r^2a^2\frac{d\theta}{ds}^2 = \frac{L^2}{a^2r^2} - \frac{l^2}{a^2r^2 sin(\theta)^2} \tag{32}$$

In the same way, substituting the above into the r equation,and choosing $\mu = \int \frac{dt}{a^2}$ as the independent variable, we finally get

$$\frac{a^2}{1-kr^2}\frac{dr}{ds}^2 + \frac{L^2}{a^2r^2} = K^2 \tag{33}$$

where K is a constant. We note that in order that r go to zero, we must have L=0, or the second term on the left will always dominate and be larger than K^2 before r gets to 0.

If we choose L=0, then the equation for the null geodesic is

$$0 = \frac{dt}{ds}^2 - \frac{a^2}{1-kr^2}dr^2 \tag{34}$$

$$\text{or} \tag{35}$$

$$\frac{dr}{dt} = \frac{\sqrt{1-kr^2}}{a} \tag{36}$$

$$\text{or} \tag{37}$$

$$\int_0^R \frac{dr}{1-kr^2} = \int_{t_1}^{t_2} \frac{dt}{a(t)} \tag{38}$$

where t_1 is the time the light ray leaves from $r = 0$ and t_2 when it arrives at $r = R$.

If we assume another null ray leaves at $t_1 + \Delta t$ where Δt is very small, and arrives at $t_2 + \delta t$, then we have

$$\int_0^R \frac{dr}{1-kr^2} = \int_{t_1+\Delta t}^{t_2+\delta t} \frac{dt}{a(t)} \tag{39}$$

Keeping only to lowest order in Δt and δt we finally get

$$\frac{\delta t}{a(t_2)} - \frac{\Delta t}{a(t_1)} = 0 \tag{40}$$

or

$$deltat = \frac{a(t_2)}{a(t_1)}\Delta t \tag{41}$$

This is the cosmological redshift.

If we assume that a particle leaves from r=0 at angles θ, ϕ those angles remain constant. Thus if there are N particles within some solid angle, there will be N particles always withing that solid angle. The area designatied by that solid angle with angles $\delta\theta$, $\delta\phi$ will be the proper distances corresponding to those angles– namely

$$\Delta A = (ar\delta\theta)(ar\sin(\theta)\delta\phi \tag{42}$$

. Ie, the surface density of those particles will be $N/\Delta A$ Since $\delta\theta$, $\delta\phi$, θ all remain consant, the density scales as $\frac{1}{a(t)r}$. If the density at a unit distance from the source is ρ_1 (ie $a(t_1)r = 1$, then the density at the observation point R at time t_2 is $\frac{\rho_1}{(a(t_2)R)^2}$. Note that this is not just the distance from the star squared if the spatial metric is not flat, since the spatial distance is $a(t)\int_0^R \frac{dr}{1-kr^2}$

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b) Show that the curve r=0 is a timelike geodesic.

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From a) L=l=K=0 is a valid geodesic. But these imply that Θ, ϕ, r are all constants is a geodesic. Since only $\frac{dt}{ds}$ is non-zero, the length squared of the tangent vector is negative (ie it is a timelike geodesic).

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3.Consider a flat, dust filled universe, for which $a(t) = a_0 t^{2/3}$. Write the metric in terms of the area coordinate R defined so that the angular part of the metric is $R^2(d\theta^2 + sin(\theta)^2 d\phi^2)$, and t, the normal cosmological time.

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$$ds^2 = -dt^2 + a(t)^2(dr^2 + r^2 d\theta^2 + r^2 sin(\theta)^2 d\phi^2)$$

The circumference of the circle at r, $\theta = \pi/2$, is $2\pi a(t) r$. Choosing $R = a(t)r$, or $\frac{r=R}{a(t)}$ we get

$$ds^2 = -dt^2 + a^2(d(\frac{R}{a})^2 + R^2 d\theta^2 + R^2 sin(\theta)^2 d\phi^2 \tag{43}$$

$$= -dt^2 + a^2(\frac{dR}{a} - R\frac{\dot{a}}{a^2}dt)^2 + R^2 sin(\theta)^2 d\phi^2 \tag{44}$$

$$= -(1 - H^2R^2)dt^2 - 2HRdRdt + dR^2 + R^2(d\theta^2 + sin(\theta)^2 d\phi^2) \tag{45}$$

where $H = H(t) = \frac{d\ln(a(t)}{dt}$ If $a = a_0 t^{2/3}$, then $H = \frac{2}{3t}$.

As an interesting aside, if $a = a_0 e^H t$, with H a constant, we can define a new t coordinate by

$$ds^2 = -(1 - H^2R^2)(dt^2 + 2\frac{HR}{1 - H^2R^2}dtdR) + dR^2 \tag{46}$$

$$+R^2(d\theta^2 + sin(\theta)^2 d\phi^2) \tag{47}$$

$$= -(1 - H^2R^2)(dt + \frac{HR}{1 - H^2R^2}dR)^2 + (1 + \frac{H^2R^2}{1 - H^2R^2})dR^2 \tag{48}$$

$$+R^2(d\theta^2 + sin(\theta)^2 d\phi^2) \tag{49}$$

Defining $\tau = t - \int \frac{HR}{1-H^2R^2}dR$, we finally have

$$ds^2 = -(1 - H^2R^2)d\tau^2 + \frac{1}{1 - H^2R^2}dR^2 + R^2(d\theta^2 + sin(\theta)^2 d\phi^2) \tag{50}$$

Note that this has the same form as the Schwartzschild metric, with a singularity at $R = \frac{1}{H}$. This metric (with H constant) is DeSitter spacetime and the singular surface is the cosmological horizon. Note again that this horizon is a coordinate singularity.

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