Physics 407-09
Assignment 5

1) Show that for the metric

$$ds^2 = (1 - \frac{2M}{r})(-dt^2 + dr^2 + r^2 d\theta^2 + r^2 sin^2(\theta) d\phi^2) \tag{1}$$

there is no deflection of light.

(solve the geodesic equations, where you can again assume that $\theta = \pi/2$ and find the equation for r as a function of ϕ. Show that the straight lines as a function of r and ϕ are exactly the same as those in flat spacetime).

It was this in part which caused Einstein to reject the Nordstrom theory, which predicted a metric such as the above.

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The geodesic equations are

$$\frac{d}{ds}\left((1-2M/r)r^2 sin(\theta)^2 \frac{d\phi}{ds}\right) = 0 \tag{2}$$

$$\frac{d}{ds}\left(i(1-2M/r)r^2 \frac{d\theta}{ds}\right) - (1-2M/r)sin(\theta)cos(\theta)\frac{d\phi}{ds}^2 = 0 \tag{3}$$

$$\frac{d}{ds}\left((1-2M/r)\frac{dt}{ds}\right) = 0 \tag{4}$$

The equation for the variation with respect to r can be replaced by the length of the geodesic

$$(1-2M/3)(-\frac{dt}{ds}^2 + \frac{dr}{ds}^2 + r^2(\frac{d\theta}{ds}^2 + sin(\theta)^2\frac{d\phi}{ds}^2) = -\mu \tag{5}$$

where $\mu = 1$ for timelike paths and 0 for lightlike.

Again we can take $\theta = \pi/2$ as a solution of the θ equation.

$$\frac{d\phi}{dt} = \frac{L}{(r^2(1-2M/r))} \tag{6}$$

$$\frac{\frac{dt}{ds} = E}{(1-2M/r)} \tag{7}$$

$$\frac{dr}{ds}^2 = \frac{E^2}{(1-2M/r)^2} - \mu/(1-2M/r) - \frac{L^2}{(r^2(1-2M/r)^2)} \tag{8}$$

Again using the first and last to write these in terms of r and ϕ instead

$$\frac{L^2}{(1-2M/r^2)r^4}\frac{dr}{d\phi}^2 = \frac{E^2}{(1-2M/r)^2} - \mu/(1-2M/r) - \frac{L^2}{(r^2(1-2M/r)^2)} \tag{9}$$

or writing in terms of $u = 1/r$ and ϕ we have

$$\frac{du}{d\phi}^2 = E^2/L^2 - (\mu/L^2)(1-2Mu) - u^2 \tag{10}$$

For $\mu = 0$, lght rays, this is exactly the same as the equation for a straight line ($u = E/Lcos(\phi - \phi_0)$ or $r\cos(\phi - \phi_0) = L/E$ which is just the equation for a straight line in $r\phi$ coordinates.

For $\mu = 1$ this is exactly the same as Newton's equations for orbits around a central mass, and has no perihelion advance.

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2) Consider the Schwartzschild metric. Do a coordinate transformation such that $t = \tau + f(r)$. Find the functions $f(r)$ such that the spatial part of the metric is flat space. (Ie, the part of the metric which does not depend on dt is just

$$dr^2 + r^2 d\theta^2 + r^2 sin^2(\theta) d\phi^2$$

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Let us ignore the angular part for now, since it does not change.

$$-(1 - 2M/r)dt^2 + \frac{dr^2}{(1 - 2M/r)} = -(1 - 2M/r)(d\tau + f'(r)dr)^2 + \frac{dr^2}{(1 - 2M/r)} \quad (11)$$

$$= -(1 - 2M/r)d\tau^2 - 2(1 - 2M/r)f'(r)drd\tau - f'^2(1 - 2M/r)dr^2 - \frac{dr^2}{(1 - 2M/r)} \quad (12)$$

where $f'(r) = \frac{df(r)}{dr}$. We want the dr^2 term to have a prefactor of 1, so we want

$$-f'^2(1 - 2M/r) + \frac{1}{(1 - 2M/r)} = 1 \quad (13)$$

$$f'^2 = \frac{2M/r}{(1 - 2M/r)^2} \quad (14)$$

Thus the metric will be

$$ds^2 = -(1 - 2M/r)d\tau^2 \pm \sqrt{\frac{2M}{r}} dr d\tau + dr^2 + r^2(d\theta^2 + sin(\theta)^2 d\phi^2 \quad (15)$$

where the $\pm$ corresponds to the different signs of the square root in finding $f'(r)$

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These metrics are called the Panlevi-Gulstrand metrics. Show that at $r = 2M$ this metric is not singular (ie does not blow up and has a well defined non-singular inverse).

Show that for the two possible signs of $f(r)$, the surface $r = 2M$ in the one metric is not the same as the surface $r = 2M$ in the other metric.

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$$t = \tau \pm \int \frac{\sqrt{2M/r}}{(1 - 2M/r)} dr = \tau \pm \int \frac{\sqrt{2Mr}}{r - 2M)} dr \quad (16)$$

$$= \tau \pm 2M \int 2y^2/overy^2 - 1 dy = \tau \pm (2M(y + \int \frac{d\xi}{cosh(\xi)}) \quad (17)$$

where $\frac{y^2=r}{2M}$ and $y = sinh(\xi)$. Taking the integral to $r = 2M$, or $y = 1$ we find that the integral diverges as $ln(r - 2M)$. Thus $r - 2M$ corresponds to $t = +\infty$ in the one case, and to $t = -\infty$ in the other case. Ie, they correspond to two totallydifferent limits.

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3. At what radius does the unstable circular orbit occur as a function of l and E? What is the innermost unstable circular orbit for arbitrary l and E? (This is in the Schwartzschild metric, and it is for massive particles.)
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The equation is

$$\frac{du}{d\phi}^2 = \frac{E^2}{L^2} - \frac{(1-2Mu)}{L^2} - u^2(1-2Mu) \tag{18}$$

For a circular orbit, both $\frac{du}{d\phi}$ and $\frac{d^2u}{d\phi^2}$ have to be zero. Thus

$$\frac{E^2}{L^2} - \frac{(1-2Mu)}{L^2} - u^2(1-2Mu) = 0 \tag{19}$$

$$\frac{2M}{L^2} - 2u + 6Mu^2 = 0 \tag{20}$$

The first can always be solved by an appropriate choice of E/L The second gives

$$u = (2 \pm \sqrt{4 - 48\frac{M^2}{L^2}})/12M \tag{21}$$

These two correspond to a stable and an unstable circular orbit. The smallest one is the stable on (minimum of the energy) since the second derivative is $2 - 12Mu$, and this is positive for small u. We get the largest value for the smaller solution if the square root is zero, in which case

$$u = \frac{1}{6M} \tag{22}$$

Thus the innermost circular orbit occurs for r=6M.

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4. Consider a galaxy of mass 10^{10} times the mass of the sun. (called the imaging galaxy) on opposite sides of which we see two images of a much more distant galaxy (the imaged galaxy). Assume that one of those images of the imaged galaxy is seen to be twice as far away from the center of the imaging galaxy as is the other (all angles are a few seconds of arc). The distant imaged galaxy has a supernova go off in it. What would be the difference in times at which we would see that supernova in the two images here on earth as a function of the distance of that imaging galaxy from the earth. (assume that the distance of the imaged galaxy is many times as far away from the earth than

is the imaging galaxy You can assume that the imaging galaxy is also very far from the earth.)

Note that you can assume that the images are further away from the center of the imaging galaxy than is the edge of the matter distribution in that imaging galaxy, and that the imaging galaxy can be represented as a spherical source of gravity.

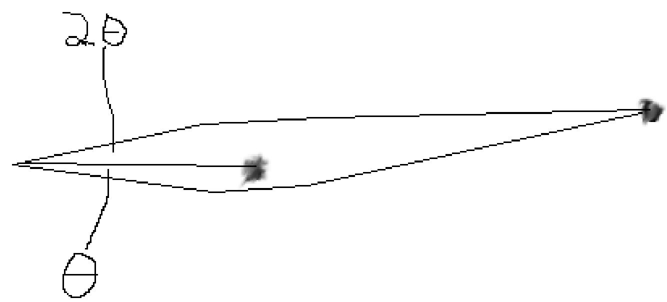

Figure 1: *Geometry of the deflection of distant galaxy light by nearby galaxy*

The time delay goes as $4Mln(r_{min})$ where r_{min} is the radius of closest approach. Thus the time difference goes as $4M(ln(r_1) - ln(2r_1)) = 4Mln(2)$ Ie the time difference does not depend on r_min but only on the mass of the galaxy. In our units GM/c^2 is a distance, so GM/c^3 is a time. We have

$$\Delta t = 4GM/c^3 ln(2) = 2.5(.5 \cdot 10^{-5} \times 10^{10})sec = 1.2 \cdot 10^5 sec. \approx 1.3 days. \quad (23)$$

This could be used backwards to find the total mass of the galaxy by measuring the time delay between the two images.

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