Physics 407-07
Assignment 5

Note: In the following D_A is to be taken to be the same thing as ∇_A in the lectures.

1.) Consider the metric

$$ds^2 = -\rho^2 d\tau^2 + d\rho^2 \tag{1}$$

i)What are the Christofel symbols for this metric?
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$$g_{\tau\tau} = -\rho^2 \tag{2}$$
$$g_{\rho\rho} = 1 \tag{3}$$

and the rest 0, so the matrix is

$$g_{ij} = \begin{pmatrix} -\rho^2 & 0 \\ 0 & 1 \end{pmatrix} \tag{4}$$

so the inverse matrix is

$$g_{ij} = \begin{pmatrix} -\frac{1}{\rho^2} & 0 \\ 0 & 1 \end{pmatrix} \tag{5}$$

or

$$g^{\tau\tau} = -\frac{1}{\rho^2} \tag{6}$$
$$g^{\rho\rho} = 1 \tag{7}$$
$$g^{\tau\rho} = g^{\rho\tau} = 0 \tag{8}$$

From

$$\Gamma^i{}_{jk} = \frac{1}{2}\sum_l g^{il}\left(\partial_j g_{lk} + \partial_k g_{jl} - \partial_l g_{jk}\right) \tag{9}$$

we see that the only term which will survive is if two of ijk are τ and one is ρ, with the ρ being the derivative.

$$\Gamma^\tau_{\tau\rho} = \Gamma^\tau_{\rho\tau} = \frac{1}{2} g^{\tau\tau}\partial_\rho g_{\tau\tau} = \frac{1}{\rho} \tag{10}$$

and

$$\Gamma^\rho_{\tau\tau} = \frac{1}{2} g^{\rho\rho}(-\partial_\rho g_{\tau\tau}) \tag{11}$$
$$= \rho \tag{12}$$

Alternatively, one could use the variational principle to get the geodesic equations

$$\delta \int g_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} ds = 0 \tag{13}$$

or

$$\frac{d}{ds}(\rho^2 \frac{d\tau}{ds}) = 0 \tag{14}$$

$$-\frac{d}{ds}(\frac{d\rho}{ds}) - \rho(\frac{d\tau}{ds})^2 = 0 \tag{15}$$

or writing in terms of the second derivatives alone

$$\frac{d^2\tau}{ds^2} + 2\rho \frac{d\tau}{ds}\frac{d\rho}{ds} = 0 \tag{16}$$

$$\frac{d^2\rho}{ds^2} + \rho(\frac{d\tau}{ds})^2 = 0 \tag{17}$$

from which we can read off

$$\Gamma^{\tau}_{\tau\rho} = \Gamma^{\tau}_{\rho\tau} = \frac{1}{\rho} \tag{18}$$

$$\Gamma^{\rho}_{\tau\tau} = \rho \tag{19}$$

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ii) What is the acceleration vector and its magnitude for the curve $\rho = \rho_0$ at various times τ?

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Note that since we are using τ as one of the coordinates, we cannot use it as the timelike path-length parameter. I will thus use s to designate that parameter, even though usually s is the pathlength parameter for spacelike curves.

The acceleration is the derivative of the velocity. In order that this be a tensor relation, the derivative must be the covariant, parallel, derivative along the curve.

$$a^A = \frac{Du^A}{Ds}$$

The components of a^A are then

$$\sum_i a^i \frac{\partial}{\partial x^i}^A = \frac{Du^i}{Ds}\frac{\partial}{\partial x^i}^A + u^i \frac{D}{Ds}\frac{\partial}{\partial x^i}^A \tag{20}$$

$$= (\frac{du^i}{ds} + u^j \Gamma^i_{jk} u^k)\frac{\partial}{\partial x^i}^A \tag{21}$$

(where I have used the summation convention, that if upper and lower indices are repeated, then that also means that they are summed over.)

Now, if the curve is $\rho = \rho_0$. then s must be chosen to be the length along the curve, which is

$$ds = \sqrt{\rho_0^2 d\tau^2} = \rho_0 d\tau \tag{22}$$

or

$$\tau = \frac{s}{\rho_0} \tag{23}$$

Thus

$$\frac{d\tau}{ds} = \frac{1}{\rho_0} \tag{24}$$

$$\frac{d^2\tau}{ds^2} = \frac{d\rho}{ds} = \frac{d^2\rho}{ds^2} = 0 \tag{25}$$

and thus

$$a^\tau = \frac{d^2\tau}{ds^2} + \frac{2}{\rho_0}\frac{d\tau}{ds}\frac{d\rho}{ds} = 0 \tag{26}$$

$$a^\rho = \frac{d^2\rho}{ds^2} + \rho_0 \frac{d\tau}{ds}^2 = \frac{1}{\rho_0} \tag{27}$$

Ie, the acceleration is constant, and has only a ρ component, which makes sense since it must be orthogonal to the velocity which has only a τ component.

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iii)If you have a field ϕ defined at all values of $\tau,\ \rho$, what is the wave equation in terms of $\tau,\ \rho$

$$g^{AB} D_A(D_B\phi) = 0 \tag{28}$$

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We looked at this field equation in class, and showed that in components it is

$$g^{AB} D_A(D_B\phi) = D_A(g^{AB} D_B\phi) = \frac{1}{\sqrt{|det(g)|}}\partial_i\sqrt{|det(g)|}g_{ij}\partial_j\phi \tag{29}$$

$$= \frac{1}{\rho}\partial_\tau\rho(\frac{-1}{\rho^2})\partial_\tau\phi + \frac{1}{\rho}\partial_\rho\rho\partial_\rho\phi \tag{30}$$

$$= -\frac{1}{\rho}\partial_\tau^2\phi + \frac{1}{\rho}\partial_\rho(\rho\partial_\rho\phi) \tag{31}$$

Note the similarity between this and the laplacian in polar coordinates.

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2.)In ordinary x,y,z,t coordinates (with metric $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$), define the antisymmetric tensor $F^{AB} = -F^{BA}$ with components (and others related to these by the antisymmetry)

$$F^{tx} = E_x \tag{32}$$
$$F^{ty} = E_y \tag{33}$$
$$F^{tz} = E_z \tag{34}$$
$$F^{xy} = B_z \tag{35}$$
$$F^{yz} = B_x \tag{36}$$
$$F^{zx} = B_y \tag{37}$$

where E_i are the usual components of the electromagnetic field, and B_i are those for the magnetic field.

i) If one has a source free Electromagetic field, show that the equations

$$D_A F^{AB} = 0 \tag{38}$$
$$D_A F_{BC} + D_B F_{CA} + D_C F_{AB} = 0 \tag{39}$$

expressed in terms of the coordinates $t, x, y.z$ are the electromagnetic field equations.

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Implicit is that we are to use the standard Minkowskian coordinates. This means that the metric is all constants, and all the christofel symbols are 0, and covariant derivaties are the same as ordinary.

Thus we can look at the various components.

$$\sum_i \partial_i F^{ti} = \partial_t F^{tt} + \partial_x F^{tx} + \partial_y F^{ty} + \partial_z F^{tz} = 0 \tag{40}$$

But $F^{tt} = 0$ since it is antisymmetric, and the others can be expressed in terms of E, so this becomes

$$\nabla \cdot \vec{E} = 0 \tag{41}$$

The next is the x component

$$\sum_i \partial_i F^{xi} = \partial_t F^{xt} + \partial_y F^{xy} + \partial_x F^{xz} \tag{42}$$
$$= -\partial_t E^x + \partial_y B_z - \partial_z B_y \tag{43}$$

and similarly for the y and z components to give

$$-\partial_t \vec{E} + \nabla \times \vec{B} = 0 \tag{44}$$

The metric is diagonal with g_{tt} being -1 and $gxx = g_{yy} = g_{zz} = 1$.This means that the components of F_{ij} will be the same as those of F^{ij} except that those with one t in ij will be of the opposite sign.

$$F_{ti} = -E_i \tag{45}$$
$$F_{xy} = B_s; \qquad F_{yz} = B_x \qquad F_{zx} = B_y \tag{46}$$

Because

$$\nabla_A F_{BC} + \nabla_B F_{CA} + \nabla_C F_{AB}$$

is completely antisymmetric, the components are non zero only if ijk are all different.

$$\partial_x F_{yx} + \partial_y F_z x + \partial_z F_{xy} \quad (47)$$

$$= \partial_x B_x + \partial_y B_y + \partial_z B_z = 0 \quad (48)$$

or

$$\vec{\nabla} \times \vec{B} = 0 \quad (49)$$

Similarly looking at the one where ijk does not include x

$$\partial_t F_{yz} + \partial_y F_{zt} + \partial_z F_{tx} \quad (50)$$

$$= \partial_t B_x + \partial_y E_z - \partial_z B_y = 0 \quad (51)$$

or

$$\partial_t \vec{B} + \vec{\nabla} \times \vec{B} = 0 \quad (52)$$

which are exactly Maxwell's equations.

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(Note– first show that for the components, the three component indicees in the second equation must all be different for the left hand side to be non-zero. Thus there are really only 4 non-trivial equations.

ii)Show that in general for an arbitrary antisymmetric $F^{AB} = -F^{BA}$, that $D_A D_B F^{AB} = 0$ Note that while the antisymmetric derivative of a scalar is assumed to be zero, you cannot make this assumption for a tensor. Instead look at the components of this tensor expression and use the properties of the Christofel symbols.

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$$D_B F^{AB} \to \partial_j F^{ij} + \Gamma^i_{kj} F^{kj} + \Gamma^j_{kj} F^{kj} \quad (53)$$

$$= \partial_j F^{ij} + \Gamma^j_{kj} F^{kj} \quad (54)$$

since F is anti symmetric in its indices and Γ is symmetric in its lower two indices.

Now this vector, $W^A = D_B F^{AB}$ has derivative

$$D_A W^A \to \partial_i W^i + \Gamma^i_{li} W^l \quad (55)$$

Thus

$$D_A D_B F^{AB} \to \partial_i(\partial_j F^{ij} + \Gamma^i_{kj} F^{kj} + \Gamma^j_{kj} F^{ik}) + \Gamma^i_{li}(\partial_j F^{lj} + \Gamma^l_{kj} F^{kj} + \Gamma^j_{kj} F^{lk}) \quad (56)$$

$$= \partial_i \partial_j F^{ij} + \partial_i(\Gamma^i_{kj} F^{kj} + \Gamma^j_{kj} F^{ik}) + \Gamma^i_{li}(\partial_j F^{lj} + \Gamma^i_{li} \Gamma^l_{kj} F^{kj} + \Gamma^j_{kj} F^{lk}) \quad (57)$$

Now the first term is zero, because for each term like $\partial_x\partial_y F^{xy}$ we have another one $\partial_y\partial_x F^{yx} = -\partial_y\partial_x F^{xy} = -\partial_x\partial_y F^{xy}$ which cancells it.

terms like $\Gamma^l_{jk}F^{jk}$ are zero because F is antisymmetric while the Christofel symbols are symmetric and thus $\Gamma^l_{kj}F^{kj} = \Gamma^l_{jk}F^{jk}$ by renaming the summation indices $k \leftrightarrow j$ and this equals $\Gamma^l_{kj}(-F^{kj} = -\Gamma^l_{kj}F^{kj}$ so the term equals its negative.

Thus we are left with

$$D_A D_B F^{AB} \rightarrow \partial_i(\Gamma^j_{kj}F^{ik}) + \Gamma^i_{li}\partial_j F^{lj} + \Gamma^i_{li}\Gamma^j_{kj}F^{lk} \tag{58}$$

In the last term the product of the two Γ are symmetric while F is antisymmetric so this term is zero. Thus

$$\partial_i(\Gamma^j_{kj}F^{ik}) + \Gamma^i_{li}\partial_j F^{lj} = (\partial_i\Gamma^j_{kj})F^{ik} + \Gamma^j_{kj}\partial_i F^{ik} + \Gamma^i_{li}\partial_j F^{lj} \tag{59}$$

The last two terms cancel (relabel $i \leftrightarrow j$ and relabel k as l and the two terms are seen to be identical except fot the reversed indices of F. Since F is antisymmetric, the two terms cancel).

Finally we need to look at

$$(\partial_i\Gamma^j_{kj}) = \partial_i\partial_k(\frac{1}{2}ln(|det\ g\ |)) \tag{60}$$

which is symmetric in i and k, and thus this term also cancels.

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iii)Find $F^A{}_A$ and $F^{AB}F_{AB}$ in terms of E and B.

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Since F is antisymmetric, $F^A_A = g_{AB}F^{AB} = g_{BA}F^{BA}$ by relabeling the contraction indicees $g_{BA}F^{BA} = (g_{AB})(-F^{AB} = -g_{AB}F^{AB}$ Thus $g_{AB}F^{AB} = -g_{AB}F^{AB}$ and thus must be zero.

Using the Minkowski metric which is diagonal, we have that $F_{ti} = g_{tt}g_{ii}F^{ti} = -F^{ti}$ for i=x,y,z. And $F_{ij} = g_{ii}g_{jj}F^{ij}$. Thus (using that $F^{tt} = F^{xx} = F^{yy} = F^{zz} = 0$)

$$F^{AB}F_{AB} = F^{tx}F_{tx} + F^{ty}F_{ty} + F^{tz}F_{tz} + F^{xt}F_{xt} + F^{xy}F_{xy} \tag{61}$$

$$+F^{xz}F_{xz} + F^{yt}F_{yt} + F^{yx}F_{yx} + F^{yz}F_{yz} + F^{zt}F_{zt} + F^{zx}F_{zx} + F^{zy}F_{zy} \tag{62}$$

$$= 2(-\sum_{i=1}^{3} F^{ti^2} + 2(F^{xy2} + F^{yz2} + F^{zx2}) \tag{63}$$

$$= 2(\vec{B}\cdot\vec{B} - \vec{E}\cdot\vec{E}) \tag{64}$$

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3. Show that the stress-energy tensor for the source free electromagnetic field

$$T^{AB} = F^{CA}F_C{}^B - \frac{1}{4}F^{CD}F_{CD}g^{AB} \tag{65}$$

is conserved by the equations of motion of the electromagnetic field. Ie, $\nabla_A T^{AB} = 0$.

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$$\nabla_A T^{AB} = \nabla_A(F^{AC}F^B{}_C) - \ldots$$
$$= (\nabla_A F^{AC})F^B{}_C + F^{AC}\nabla_A F \ldots$$
$$= F^{AC}\nabla_A(g^{BX}F \ldots$$
$$\text{bythefirstmaxwellequation} \ldots$$
$$= F^{AC}g^{BX}(-\nabla_X F_{BC} - \nabla_B F_{CX}) - \frac{1}{2}g^{AB}g^{CX}g^{DY}F_{XY}\nabla_A F_{CD} \quad bythesecondMaxwellequationandbyalterin\ldots$$
$$= F^{AC}g^{BX}(-\nabla_X F_{BC} + \nabla_B F_{XC}) - \frac{1}{2}g^{AB}F^{CD}(-\nabla_C F_{DA} \ldots$$
$$= 0 + -\frac{1}{2}g^{AB}F^{CD}(-\nabla_C F_{DA} + \nabla_D F_{CA} \quad byrelabelingofindicesBX \ldots$$
$$= 0 \quad \text{by}\ldots$$

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4. The relativistic Lorentz force law for a particle of mass m and charge e can be written as

$$m\frac{Du^A}{D\tau} = eu_B F^{AB} \tag{74}$$

Show that this preserves the length of the vector u^A as it should. Show that this gives the usual force law of and electric and magnetic field on a charged particle in the non-relativistic limit in the usual flat spacetime metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{75}$$

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$$mg_{AC}u^C\frac{Du^A}{D\tau} = eu_A u_B F^{AB} = -eu_A u_B F^{BA} = -eu_B u_A F^{AB} \quad \text{byrelabeling}AB \leftrightarrow BA\text{ontheRHS} \tag{76}$$
$$= 0 \tag{77}$$

by the antisymmetry of F. But

$$\frac{D}{D\tau}(g_{AB}u^A u^B) = g_{AB}u^B\frac{Du^A}{D\tau} + g_{AB}u^A\frac{Du^B}{D\tau} \quad \text{byproductruleandzeroderivativeofg} \tag{78}$$
$$= 2u_A\frac{Du^A}{D\tau} \tag{79}$$

which is 0 if the length squared of the velocity vector u is constant.

In the Minkowski frame where the Christofel symbols are zero, this becomes

$$du^t = overd\tau = \frac{e}{m} u_x E^x + u_y E^y + u_z E^z = u^t \frac{e}{m} \vec{v} \cdot \vec{E} \tag{80}$$

$$\tag{81}$$

using $u^x = v^x u^t$ Recalling that $\frac{d}{d\tau} = \frac{dt}{d\tau}\frac{d}{dt} = u^t \frac{d}{dt}$ this becomes

$$\frac{du^t}{dt} = \frac{e}{m} \vec{v} \cdot \vec{E}$$

which if mu^t is interpreted as the energy of the particle is just the statement that the rate of change of energy is the work done by the E field/

$$\frac{du^x}{d\tau} = \frac{e}{m} - u^t F^{xt} + u^y F^{xy} + u^z F^{xz} \tag{82}$$

$$= \frac{e}{m} u^t E^x + u^y B^z - u^z B^y \tag{83}$$

or again

$$\frac{du^x}{dt} = \frac{e}{m} E^x + (\vec{v} \times \vec{B})^x$$

or

$$\frac{d(u^t \vec{v})}{dt} = \frac{e}{m} \vec{E} + \vec{v} \times \vec{B}$$

which in the non-relativistic limit where $u^t \approx 1$ gives us the Lorentz force law for electromagnetism.

(If we assume that the momentum is $m\vec{u}$ instead of $m\vec{v}$ then we see that this force law is exactly the Lorentz force law in all cases, not just the non-relativistic limit)

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