Physics 407-07
Assignment 1

1. Given a function $f(p)$ and a set of coordinates $x^i(p)$ show that the two functions

$$f(p) \tag{1}$$

$$\sum_i \frac{\partial f(p(\mathbf{x}))}{\partial x^i}|_{p_0}(x^i(p) - x^i(p_0)) \tag{2}$$

have the same cotangent vector at the point p_0. (p(x) is the point p in the space corresponding to the coordinates $\mathbf{x}$. Those partial derivatives are evaluated at the point p_0. Since $\frac{\partial f(p(\mathbf{x}))}{\partial x^i}|_{p_0}$ are constants, by the definition of the sum of cotangent vectors, this means that

$$df_A = \sum_i \frac{\partial f(p(\mathbf{x}))}{\partial x^i}|_{p_0} dx^i_A \tag{3}$$

2. Show that if x^i and $\tilde{x}^i$ are two different coordinates , and $\gamma(\lambda)$ and $\gamma'(\lambda)$ are two different curves through the point p_0 with the point p_0 corresponding to the same value, $\lambda = 0$ in both cases, that the two curves defined by

$$\Gamma(\lambda) = p(x^i(\gamma(\lambda)) + x^i(\gamma'(\lambda)) - x^i(p_0)) \tag{4}$$

$$\tilde{\Gamma}(\lambda) = p(\tilde{x}^i(\gamma(\lambda)) + \tilde{x}^i(\gamma'(\lambda)) - \tilde{x}^i(p_0)) \tag{5}$$

have the same tangent vector at the point p_0

This shows that while the definition of the addition of two tangent vectors is defined via coordinates, the sum tangent vector thus defined does not depend on which coordinates we use, although the curves Γ and $\tilde{\Gamma}$ are in general different.

As an example, consider the two curves in two dimensions with coordinates x,y and r, θ

$$\gamma : \tag{6}$$

$$x = \lambda \tag{7}$$

$$y = 1 \tag{8}$$

$$\gamma' : \tag{9}$$

$$y = 1 + 2\lambda \tag{10}$$

$$x = 0 \tag{11}$$

Now write those same two curves in terms of the coordinates r, θ where

$$x = r\cos(theta) y = r\sin(theta) \tag{12}$$

Show that the sum curve $\Gamma(\lambda)$ defined in the two coordinate systems differ, but that at the point $\lambda = 0$ their tangent vectors do not.

3. Assume that ${H^A}_B$, ${{L_A}^B}_C$ and M_{AB} are tensors, and f, g are functions. Which of the following are tensors and why?

i)${Q_A}^B = {H^B}_A$

ii)$R = {H^A}_A$

iii)$T^D_{ABC} = {H^D}_A M_{BC}$

iv)$T^D_{ABC} = {H^D}_A + M_{BC}$

v) $R^A = {{L_B}^A}_B$

vi) $S_A = {{L_A}^B}_B - {{L_B}^B}_A$

vii) $T_A = \nabla_B {H^B}_A$

Is $\partial_i {H^j}_k$ the component of a tensor?

What are the components of expressed in terms of partial derivatives, Christofel symbols?

$$\nabla_A {H^A}_B \tag{13}$$

4. Given coordinates $r, theta$, what are the tangent vectors to the curves defined by the coordinate conditions expressed in terms of $\frac{\partial}{\partial r}^A$ and $\frac{\partial}{\partial \theta}^A$

$$r(\lambda) = r_0 \tag{14}$$
$$\theta(\lambda) = \lambda \tag{15}$$

$$r(\lambda) = \lambda \tag{16}$$
$$\theta(\lambda) = 5 * \lambda \tag{17}$$

$$r(\lambda) = 10\lambda \tag{18}$$
$$\theta(\lambda) = 50 * \lambda \tag{19}$$

What is the cotangent vector of the following functions

$$f(r, \theta) = r^2 \tag{20}$$

$$f(r, theta) = r^2 + \theta^2 \tag{21}$$

In each case find the lengths of these various vectors for each point at which they are defined if the metric is given by a)

$$ds^2 = dr^2 + d\theta^2 \tag{22}$$

and

$$ds^2 = dr^2 + r^2 d\theta^2 \tag{23}$$

5) What are all the components of the Christofel symbols for the metric

$$ds^2 = \frac{1}{r}dr^2 + rd\theta^2 \tag{24}$$

and for

$$ds^2 = \frac{1}{r}dr^2 - rdt^2 \tag{25}$$